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TRANSMISSION LINE THEORY

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TRANSMISSION LINE THEORY

AND SOME RELATED TOPICS

BY

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PREFACE

In preparing this treatise on Transmission Line Theory the authors have stressed the fundamental mathematical aspects of the subject, they have endeavoured to make use of physical conceptions wherever possible, and they have kept in mind the needs of the engineer. It is believed that the book will be very useful to the independent student and very satisfactory for class-room use.

In a hasty examination of this book the following points deserve special attention:

A unique discussion of transmission line transients (for an ideal line with zero losses) is given in Chapter I, and a fairly complete discussion of transients in circuits containing lumped L and C is given in Appendix C.

The developement of the well-known hyperbolic theory of the steady state of the a.-c. line is of the utmost importance to the engineer, and, as given in Chapter III, the conceptual use of the decaying wave-train goes far towards making the hyperbolic theory clearly intelligible.

The circle diagram of the transmission line is fully discussed in Chapter IV so that the engineer may, with a minimum effort, reach an understanding command over this important device for making transmission line calculations.

Chapter V is believed by the authors to be the only simple exposition of wave-filter theory which leads to practical and practicable methods of wave-filter design.

The discussion of Fourier's series in Chapter VI is a

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careful revision, with some additions, of a chapter in a previous book* concerning which an English reviewer expressed the opinion that it was "The best elementary discussion of the Fourier analysis to be found in the English language."

Some extremely interesting and little-known wave-motion phenomena are discussed in Article 18 of Chapter III (page 94), and the reader may be interested to know that in a many-sectioned high-pass wave-filter one would, if the waves were visible, see wave-forms traveling from the receiver towards the generator when energy is traveling from generator towards the receiver. That is to say, in the many-sectioned high-pass filter the group velocity is not only different from form velocity in numerical value, but it is opposite to form velocity in direction of travel!

An extremely simple and interesting discussion of wave-distortion and of the effects of "line loading" is given in Articles 22–25 of Chapter I (pages 35–42).

THE AUTHORS

May 27, 1926

* Franklin, MacNutt and Charles' *Calculus*.

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PART I

TRANSMISSION LINE

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TRANSMISSION LINE THEORY

CHAPTER I

NON-PERIODIC WAVES ON A TRANSMISSION LINE HAVING ZERO LOSSES

1. The principle of superposition. A property of linear differential equations. A principle of extremely wide application in physics is the so-called principle of superposition. This principle applies in all cases where cause and effect are simply proportional to each other, but from the physical point of view a general statement of the principle is hardly possible. Therefore a few simple examples must suffice. (1) A person at A in Fig. 1 can see lamp No. 1 and another person at B can see lamp No. 2 *at the same time.* This means that the two beams of light a and b travel through the region R at the same time, and that each beam behaves as if it were traveling through the region alone. (2) T wo sounds travel through the same body of air simultaneously, each sound traveling as if it were alone (except when the sounds are very intense). (3) Two systems of water waves travel over the same part of the surface of a pond simultaneously, each system behaving as if it were alone (except when the water waves are very steep and high). (4) Two messages*

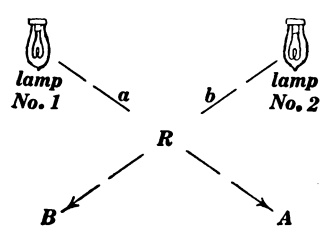

Fig. 1.

* Indeed any number of distinct messages can travel over a telegraph wire in either direction or in both directions simultaneously. The only limiting feature in multiplex telegraphy or telephony is the design of the sending and receiving apparatus. In each of the above examples the word *two* means *two or more.*

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can travel over a telegraph wire simultaneously and not get mixed up. (5) Two forces F and G exerted simultaneously on an elastic structure produce an effect which is the sum of the effects which would be produced by the forces separately, provided the strain of the structure is not carried beyond the elastic limit.

All of the effects in physics which are superposable—and this includes by far the greater portion of the effects in mechanics, heat, electricity and magnetism, light and sound, and chemistry—are expressible in terms of linear differential equations, and the principle of superposition is a clearly defined property of such equations as follows: *If y is a function of x which satisfies a linear differential equation, and if z is another function of x which satisfies the same differential equation, then $(y + z)$ is a function of x which satisfies the differential equation.*

This proposition is true for both ordinary and partial linear differential equations, and although nearly all of the superposable effects in physics and chemistry are expressed in terms of partial differential equations, it is sufficient to prove the proposition for the ordinary linear differential equation because the proof is nearly the same for ordinary and for partial linear differential equations.

Given the linear differential equation

$$u + A\frac{du}{dx} + B\frac{d^2u}{dx^2} = 0 \quad \text{.....(i)}$$

in which A and B are constants. Let y be a function of x which satisfies this equation; then

$$y + A\frac{dy}{dx} + B\frac{d^2y}{dx^2} = 0 \quad \text{.....(ii)}$$

Let z be a function of x which satisfies equation (i); then

$$z + A\frac{dz}{dx} + B\frac{d^2z}{dx^2} = 0 \quad \text{.....(iii)}$$

Now

$$\frac{d(y+z)}{dx} = \frac{dy}{dx} + \frac{dz}{dx} \quad \text{and} \quad \frac{d^2(y+z)}{dx^2} = \frac{d^2y}{dx^2} + \frac{d^2z}{dx^2}$$

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Therefore, adding equations (ii) and (iii) we get

$$(y + z) + A\frac{d(y + z)}{dx} + B\frac{d^2(y + z)}{dx^2} = 0$$

which, being identical to equation (i) in form, shows that $(y + z)$ is a function which satisfies equation (i).

2. Undetermined constants in the solution of ordinary differential equations. It is sufficient, perhaps, to illustrate by two simple examples.

(1) Consider the simple ordinary differential equation

$$\frac{dy}{dt} = a$$

where a is a constant. This equation means that the rate of change of y is constant and equal to a. Therefore the increase of y during time t is at, and the value of y at the instant t is

$$y = at + C$$

where C is the initial value of y, the value of y when $t = 0$.

(2) Consider the ordinary differential equation

$$\frac{d^2y}{dt^2} = a$$

where a is a constant. Then

$$\frac{dy}{dt} = at + B \quad \text{and} \quad y = \tfrac{1}{2}at^2 + Bt + C$$

where B is the value of $\frac{dy}{dt}$ when $t = 0$ and C is the value of y when $t = 0$.

3. Undetermined functions in the solution of a partial differential equations. It is sufficient, perhaps, to illustrate by several simple examples.

(1) A boy saves money at the constant rate of \$5 per year, beginning at 14 years of age. Integrating with respect to b (the boy), we find that the boy has \$35 when he comes of age; but does he? Something surely depends on the boy's father, and we may represent by $F(m)$, a function of m, the money that the boy's father has saved for him so that the amount of money the boy has when he comes of age will be \$35 $+ F(m)$; b and m are independent variables, let us say, and a "constant

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of integration" with respect to b turns out to be a function of m.

(2) Concerning a hill it is known, only that its slope, $\frac{\delta y}{\delta x}$, in the direction of the x-axis is constant and equal to a, so that

$$\frac{\delta y}{\delta x} = a$$

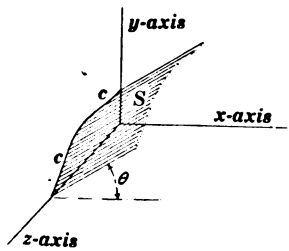

Fig. 2.

Before the complete hill or surface can be constructed from this differential equation *an arbitrary starting curve cc* (Fig. 2), must be chosen. Let $y = F(z)$ be the equation to this curve cc, then $F(z)$ is the height of the hill above the point z on the z-axis, and $ax + F(z)$ is the height of the hill above the point (x, z) in the xz-plane. That is to say, the integration of $\frac{\delta y}{\delta x} = a$ gives

$$y = ax + F(z)$$

Proposition. The general solution of an ordinary differential equation of the nth order contains n unknown constants of integration. The general solution of a partial differential equation of the nth order contains n unknown functions of integration. No attempt is here made to prove this proposition in general.

4. Example showing the use of the principle of superposition in setting up the general solution of an ordinary linear differential equation. An electric circuit consists of an inductance L, a resistance R, and a condenser of capacity C all connected in series. Let us suppose the condenser to be charged and the circuit suddenly closed. Under these conditions the condenser will discharge through the circuit. Let q be the charge on the condenser at time t. Then $\frac{dq}{dt}$ is the current in the circuit and $\frac{d^2q}{dt^2}$ is the rate of growth of the current. Therefore q/C

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is the voltage which is holding the charge q on the condenser (the voltage across the condenser terminals), $R\frac{dq}{dt}$ is the voltage being used to overcome resistance, and $L\frac{d^2q}{dt^2}$ is the voltage that is causing the current to grow. But no outside impressed voltage is acting on the circuit so that

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}\cdot q = 0 \quad \text{.....(i)}$$

This is an ordinary linear differential equation with constant coefficients, and a solution of such a differential equation can always be obtained of the form

$$q = M\epsilon^{bt} \quad \text{.....(ii)}$$

where M and b are constants. Now if (ii) is a solution of (i) it must, of course, satisfy (i). Therefore let us find expressions for $\frac{dq}{dt}$ and for $\frac{d^2q}{dt^2}$ by differentiating (ii) and substitute these expressions in (i). If we do this we find that $M\epsilon^{bt}$ will cancel out and give

$$Lb^2 + Rb + \frac{1}{C} = 0 \quad \text{.....(iii)}$$

and this equation determines the value b must have to make (ii) a solution of (i); indeed (iii) gives two values of b which we will designate as b_1 and b_2. Therefore

$$q = M_1\epsilon^{b_1t} \quad \text{and} \quad q = M_2\epsilon^{b_2t} \quad \text{(iv) and (v)}$$

are two* distinct solutions of (i) and their sum is also a solution. Therefore

$$q = M_1\epsilon^{b_1t} + M_2\epsilon^{b_2t} \quad \text{.....(vi)}$$

is a solution of (i) and since it contains two integration constants M_1 and M_2 it is the general solution.

This example shows the use of the principle of superposition in the setting up of the general solution of an ordinary linear differential equation. The principle of superposition is very

* When equation (iii) has two equal roots equation (vi) degenerates to a particular solution of (i). The general solution for this case is given in Appendix C.

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much more extensively used, however, in the setting up of solutions of partial linear differential equations. The method is to select a number (sometimes an infinite series) of elementary solutions of a partial linear differential equation and superpose or add these elementary solutions so as to get a resultant solution which satisfies all of the conditions of a problem. This method is used throughout the following discussion of electric waves on a transmission line and it is used in Chapter VI in the solution of problems of mechanical wave motion.

The problem of the plucked string is solved by the Fourier method, as set forth in Chapter VI, by finding and superposing a series of simple modes of oscillation of the string, the so-called

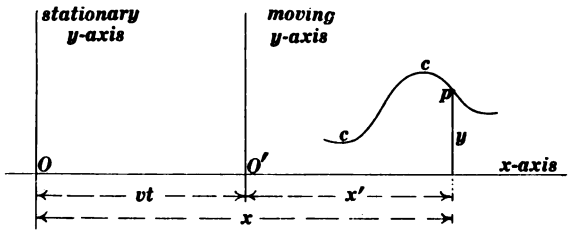

Fig. 3.

harmonics of the string. Similarly cylindrical and spherical harmonics are simple or elementary solutions of important partial linear differential equations which are superposed to give resultant solutions which satisfy given conditions.

Remark. A discussion of transient conditions in ordinary electrical circuits (in circuits *not* containing distributed capacity and inductance) is given in Appendix C.

5. Equations of a traveling curve. Let cc (Fig. 3), be any given curve which is traveling towards the right at velocity v, and let

$$y = f(x') \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

be the equation to cc referred to the x-axis which is shown in Fig. 3 and to the y-axis which passes through the point O' and travels with the curve cc. It is required to find the

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equation to the traveling curve cc referred to the x-axis which is shown and to the stationary y-axis which passes through the point O.

The abscissa of O' referred to O is vt so that $x' = x - vt$; and if we substitute this value for x' in equation (i) we get

$$y = f(x - vt) \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

which is the required equation to the traveling curve cc referred to stationary axes.

Similarly, it is easily shown that

$$y = F(x + vt) \quad \ldots\ldots\ldots\ldots\ldots \text{(iii)}$$

is the equation to a curve which is traveling towards the left in Fig. 3 at velocity v, referred to stationary axes.

6. Differential equations of travel. It is desired to set up a differential equation of which $y = f(x - vt)$ is the complete solution, another differential equation of which $y = F(x + vt)$ is the complete solution, and a third differential equation of which $y = f(x - vt)$ and $y = F(x + vt)$ are particular solutions. To get a differential equation of which $y = f(x - vt)$ is the complete solution find expressions for $\frac{\delta y}{\delta x}$ and for $\frac{\delta y}{\delta t}$ and combine them to eliminate the unknown function. Differentiating $y = f(x - vt)$ as suggested we get

$$\frac{\delta y}{\delta x} = f'(x - vt) \quad \ldots\ldots\ldots\ldots\ldots \text{(iv)}$$

and

$$\frac{\delta y}{\delta t} = -vf'(x - vt) \quad \ldots\ldots\ldots\ldots \text{(v)}$$

Where $f'(x - vt)$ denotes the derivative of $f(x - vt)$ with respect to the argument $(x - vt)$. Therefore, substituting the value of $f'(x - vt)$, namely, $\frac{\delta y}{\delta x}$, in (v), we get

$$\frac{\delta y}{\delta t} = -v\frac{\delta y}{\delta x} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(vi)}$$

which is a partial differential equation of which $y = f(x - vt)$ is the complete solution, f denoting any function whatever. Equation (vi) may therefore be called the differential equation of travel towards the right.

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Proceeding in a similar manner, equation (iii) of Art. 5 gives

$$\frac{\delta y}{\delta t} = +v\frac{\delta y}{\delta x} \quad \text{(vii)}$$

which may be called the differential equation of travel towards the left.

If we differentiate $y = f(x - vt)$ twice with respect to x and twice with respect to t we get

$$\frac{\delta^2 y}{\delta x^2} = f''(x - vt) \quad \text{(viii)}$$

and

$$\frac{\delta^2 y}{\delta t^2} = +v^2 f''(x - vt) \quad \text{(ix)}$$

so that we get

$$\frac{\delta^2 y}{\delta t^2} = v^2\frac{\delta^2 y}{\delta x^2} \quad \text{(x)}$$

and this same second order differential equation is obtained if we differentiate $y = F(x + vt)$ twice with respect to x and twice with respect to t. Therefore $y = f(x - vt)$ and $y = F(x + vt)$ *are both particular solutions of equation* (x), and, according to the principle of superposition, these two particular solutions can be added to get a solution, namely,

$$y = f(x - vt) + F(x + vt) \quad \text{(xi)}$$

and this is the complete solution of equation (x) because it contains two undetermined functions.

Remark. The notation $\frac{\delta y}{\delta x}$ and $\frac{\delta y}{\delta t}$ is used in this article because we have two independent variables x and t, and all the differential equations of this article are partial differential equations. This notation is not used hereafter, however, because partial differential equations can always be recognized as such from the fact that they contain two or more independent variables. Thus in the following article x and t are independent variables, and, of course, all of the differential equations in the following article are partial differential equations and are easily recognized as such without the use of the conventional " δ " notation.

7. Differential equations of electrical wave motion on a transmission line. The horizontal lines in Fig. 4 represent

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the two wires of a transmission line and the heavy portions of the lines represent an element of the transmission line of length Δx. Whenever we speak of the voltage e at a point on the transmission line we mean the voltage between the wires, and this voltage is considered as positive when the upper wire is positively charged and the lower wire is negatively charged. Whenever we speak of the current i at a point on the transmission line we mean an outgoing current of this value in one wire and an equal return current in the other wire, and i is considered as positive when (in Fig. 4) the current in the upper wire flows towards the right and the equal current in the lower wire flows towards the left.

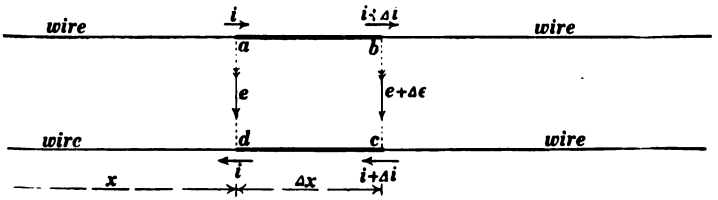

Fig. 4.

Let R be the combined resistance of the two wires in unit length of the line, let G be the leakage conductance (reciprocal of insulation resistance) between the two wires of unit length of the line, let L be the inductance of unit length of the two-wire line, and let C be the wire-to-wire capacity of unit length of the two-wire line. The quantities R, G, L and C are hereafter spoken of as *resistance, leakance, inductance* and *capacitance* of the line per unit length.

Let e be the voltage across the line at ad (Fig. 4), let $e + \Delta e$ be the voltage across the line at bc, let i be the current in the line at ad, and let $i + \Delta i$ be the current in the line at bc. The arrows in Fig. 4 represent the *positive directions* of voltage and current.

Let us speak of the change of current from a to b as a decrease. This decrease is evidently equal to the current which is leaking across from ab to cd ($= Gc \cdot \Delta x$) plus the

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current which is increasing the charge on ab $\left(= C\cdot\frac{de}{dt}\cdot\Delta x\right)$. Therefore, using $-\Delta i$ for this decrease of current, we have

$$-\Delta i = Ge\cdot\Delta x + C\cdot\frac{de}{dt}\cdot\Delta x$$

whence we get

$$\frac{di}{dx} = -Ge - C\frac{de}{dt} \quad \text{.....(i)}$$

Let us speak of the change of voltage from ad to bc as a decrease or drop of voltage. This drop is evidently equal to the resistance drop $(= Ri\cdot\Delta x)$ plus the inductance drop $\left(= L\cdot\frac{di}{dt}\cdot\Delta x\right)$. Therefore, using $-\Delta e$ for the total voltage drop, we have

$$-\Delta e = Ri\cdot\Delta x + L\cdot\frac{di}{dt}\cdot\Delta x$$

whence we get

$$\frac{de}{dx} = -Ri - L\frac{di}{dt} \quad \text{.....(ii)}$$

Equations (i) and (ii) both contain the two dependent variables e and i, and it is necessary to eliminate e or i from one of these equations by using the other equation. Thus, if we differentiate (i) with respect to t and (ii) with respect to x, we get

$$\frac{d^2i}{dxdt} = -G\frac{de}{dt} - C\frac{d^2e}{dt^2} \quad \text{.....(iii)}$$

and

$$\frac{d^2e}{dx^2} = -R\frac{di}{dx} - L\frac{d^2i}{dxdt} \quad \text{.....(iv)}$$

whence, using the value of $\frac{d^2i}{dxdt}$ from (iii) in (iv), and using the value of $\frac{di}{dx}$ from (i), we get

$$\frac{d^2e}{dx^2} - RGe - (RC + GL)\frac{de}{dt} - LC\frac{d^2e}{dt^2} = 0 \quad \text{.....(v)}$$

and a precisely similar equation involving i may be obtained by differentiating equation (i) with respect to x and (ii) with respect to t and using the value of $\frac{de}{dx}$ from (ii), namely,

$$\frac{d^2i}{dx^2} - RGi - (RC + GL)\frac{di}{dt} - LC\frac{d^2i}{dt^2} = 0 \quad \text{.....(vi)}$$

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These two differential equations (v) and (vi) are very difficult to solve except in the special cases in which e (or i) is expressible as a product XT where X is a function of x and T is a function of t, and the only solution of importance which is expressible in this form is the steady state solution which is discussed in Chapter III.

The remainder of this chapter is devoted to the simple ideal case in which $R = 0$ and $G = 0$, that is to say, the case of an ideal transmission line which is perfectly insulated and of which the wires have zero resistance. In this case the above equations (i), (ii), (v) and (vi) reduce, respectively, to

$$\frac{di}{dx} = -C\frac{de}{dt} \quad \text{(vii)}$$

$$\frac{de}{dx} = -L\frac{di}{dt} \quad \text{(viii)}$$

$$\frac{d^2e}{dt^2} = v^2\frac{d^2e}{dx^2} \quad \text{(ix)}$$

and

$$\frac{d^2i}{dt^2} = v^2\frac{d^2i}{dx^2} \quad \text{(x)}$$

where

$$v = \sqrt{1/LC} \quad \text{(xi)}$$

Let us first consider equation (ix). Two important special solutions of this differential equation, according to Art. 6, are

$$e = f(x - vt) \quad \text{(xii)}$$

and

$$e = F(x + vt) \quad \text{(xiii)}$$

and the sum of these two special solutions gives the general solution of (ix), but the conceptual development of the matter under consideration is greatly facilitated by thinking of these two special solutions separately.

Now to say that e is a function of x is equivalent to saying that there is a definite voltage distribution over the line, and equation (xii) represents such a voltage distribution which is traveling towards the right in Fig. 4. Similarly, equation (xiii) represents a voltage distribution which is traveling towards the left in Fig. 4. The velocity of travel in each case being given by equation (xi).

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Such a traveling voltage distribution is not, however, a complete electric wave. *A complete electric wave on a transmission line consists of a voltage distribution and a current distribution which travel along together and mutually sustain each other.* It is necessary therefore to determine the current distribution which must be associated with any given traveling voltage distribution to constitute a complete wave. This can be done by using equations (vii) and (viii), but it is very much more instructive to determine the associated current distribution from elementary physical considerations, as in the following articles.

8. Pure wave traveling towards the right in Fig. 4. Imagine voltage to be distributed over a transmission line

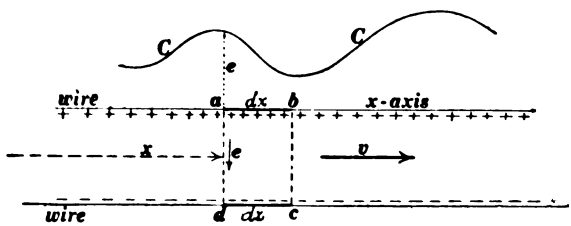

Fig. 5.

so that the voltage across the line at each point is represented by the ordinate at that point of any arbitrary curve CC as shown in Fig. 5. Then if the curve CC is traveling to the right at velocity v we will have a complete picture of equation (xii) of Art. 7—a traveling voltage distribution.

The capacity of the line element dx is $C \cdot dx$, the positive charge on ab is $eC \cdot dx$, and this charge must flow past the point b during the time dx/v required for the voltage distribution to travel the distance dx. Therefore, dividing $eC \cdot dx$ by dx/v we get the current i in the wire ab, namely,

$$i = Cev \quad \text{(i)}$$

or, using $1/\sqrt{LC}$ for v, we get

$$i = +\sqrt{C/L} \cdot e$$

or

$$e/i = +\sqrt{L/C} \quad \text{(ii)}$$

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Therefore a complete wave (a pure wave) traveling towards the right in Fig. 4 consists of similar voltage and current distributions traveling along together, the ratio e/i at each point of the line being equal to $+\sqrt{L/C}$.

9. Pure wave traveling towards the left in Fig. 4. Imagine the curve CC in Fig. 5 to be traveling towards the left at velocity v, then we have a complete picture of equation (xiii) of Art. 7—a voltage distribution traveling towards the left, and it is evident that the current in the wire ab must be in what we have agreed to consider as the negative direction, so that equation (ii) of Art. 8 becomes

$$e/i = -\sqrt{L/C} \quad \text{(i)}$$

Therefore a complete wave (a pure wave) traveling towards the left in Fig. 4 consists of similar voltage and current distributions traveling along together, the ratio e/i at each point of the line being equal to $-\sqrt{L/C}$.

10. The simplest point of view concerning electric waves on a transmission line. There is nothing in the discussion of Fig. 5 which requires v in equation (i) of Art. 8 to be equal to $\pm\sqrt{1/LC}$ and therefore equation (i) of Art. 8 may be interpreted to mean that any voltage distribution traveling at any velocity v on a transmission line would generate a current distribution which would travel along with the voltage distribution, and the current at each point on the line would be given by

$$i = Cev \quad \text{(i)}$$

This equation determines the current which must flow at each point of the upper wire to carry forwards the positive charges on the upper wire (or the opposite current in the lower wire to carry forwards the negative charge on the lower wire) as the voltage distribution travels along. Of course, i is reversed if either e or v is reversed.

Similarly, if we imagine any arbitrary current distribution to travel at velocity v along the line in Fig. 6, then the magnetic flux between the wires as it travels along with the imagined

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current distribution will induce a voltage e across the line at each point such that

$$e = Liv \quad \text{.....(ii)}$$

where i is the current in the line at the point.

This equation may be established as follows: Let the ordinate i of the traveling curve CC in Fig. 6 represent the current in the line at the point ad. The inductance of the line element is $L \cdot dx$, the magnetic flux $\Delta\Phi$* across the shaded area is $iL \cdot dx$, this flux travels forwards with the imagined current distribution and sweeps across the line ad in dx/v of a second, and therefore the rate at which magnetic flux sweeps across ad is equal to $iL \cdot dx$ *divided by* (dx/v) which gives Liv. But the induced electromotive force e across the transmission line at ad is equal to the rate at which magnetic flux sweeps across the geometric line ad, so that $e = Liv$.

Let us now consider the question "What does it mean physically to look upon equations (i) and (ii) as simultaneous equations?" The answer to this question is that v must refer to the same velocity, e to the same voltage and i to the same current in both equations; and this means that the traveling voltage distribution e which is producing the current distribution i (which is, of course, also traveling) is the voltage distribution which is in turn being produced by he traveling current distribution i. We thus see the meaning of the statement that a complete wave (a pure wave, so-called) consists of a voltage distribution and a current distribution traveling along together and mutually sustaining each other.

Dividing equation (ii) by equation (i) we get $e^2/i^2 = L/C$ which can be shown to give equation (ii) of Art. 8 when v is positive (towards the right) and to give equation (i) of Art. 9 when v is negative.

Solving equations (i) and (ii) for v, we get

$$v = \sqrt{\frac{1}{LC}}$$

* In any equation involving magnetic flux; abamperes, abvolts, abhenrys, etc., must be used, but units of the practical system can be used in equations (i) and (ii).

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which is the same as equation (xi) of Art. 7, and this equation shows at what velocity a voltage distribution and a current distribution must travel to mutually sustain each other.

Eliminating v from equations (i) and (ii), the resulting equation $e^2/i^2 = L/C$ can be thrown into the form

$$\tfrac{1}{2}Ce^2 = \tfrac{1}{2}Li^2 \quad \text{(iii)}$$

which shows that in a complete wave the electric energy per unit length of the line ($\frac{1}{2}Ce^2$) is at each point equal to the magnetic energy per unit length of the line ($\frac{1}{2}Li^2$).

11. Two important non-wave solutions of equations (ix) and (x) of Art. 7. It is evident that a uniform constant voltage e distributed over a transmission line is a solution

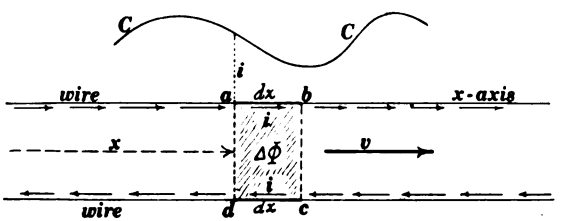

Fig. 6.

of equation (ix) of Art. 7, and this voltage distribution can exist independently of any current distribution.

It is also evident that a uniform distribution of constant current i over a transmission line is a solution of equation (x) of Art. 7, and this current distribution is independent of any voltage distribution.

It must not be forgotten that we are considering the ideal case where $R = 0$ and $G = 0$.

The use of these two extremely simple non-wave solutions of equations (ix) and (x) of Art. 7 will appear later.

12. The impure wave and its resolution into two oppositely traveling pure waves. When voltage and current at each point of a transmission line satisfy equation (ii) of Art. 8 or equation (i) of Art. 9 we have a pure wave. When neither

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of these equations is satisfied we have what is called an *impure wave.*

An impure wave immediately breaks up into, or resolves itself into, two oppositely traveling pure waves. This is equivalent to saying that if we have any voltage distribution whatever and any current distribution whatever along a transmission line (the two distributions being thought of standing still at the instant for which they are specified), then such a state of things can be resolved into two oppositely traveling pure waves [into two solutions of the wave equations (ix) and (x) of Art. 7]. This proposition is so simple when it is understood that mere examples constitute a sufficient proof.

Example 1. Imagine two wires *ad* and *bc* of zero resistance

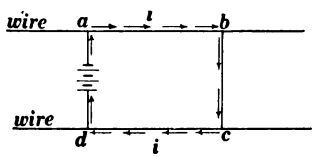

Fig. 7.

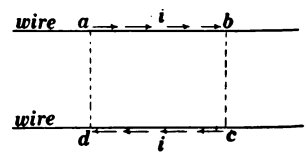

Fig. 8.

to be bridged across a transmission line as shown in Fig. 7 with a battery arranged to set up a current i in the portion *abcd* of the line, and let us imagine the two bridging wires and the battery to be instantly removed so as to leave the current i in the portion *abcd* of the line. What would happen?

The localized current distribution, which is shown in Fig. 8, would break up into two oppositely traveling pure waves. Let these oppositely traveling pure waves be symbolized by the heavy arrows A and B in Fig. 9; let e' and i' be voltage and current in wave A, and let e'' and i'' be voltage and current in wave B.

At the instant of removal of the bridging wires in Fig. 7 the simple fact is that there is current i and zero voltage over the whole of the portion *abcd* of the line, and therefore we must have

$$i' + i'' = i \quad \text{(i)}$$

and

$$e' + e'' = 0 \quad \text{(ii)}$$

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Furthermore, from Arts. 8 and 9, we must have

$$\frac{e'}{i'} = +\sqrt{\frac{L}{C}} \quad \text{.....(iii)}$$

and

$$\frac{e''}{i''} = -\sqrt{\frac{L}{C}} \quad \text{.....(iv)}$$

and these four equations determine i', e', i'' and e'', the line constants L and C and the current i being known. If we put a for $\sqrt{L/C}$ we have, in fact

$$i' = i'' = \tfrac{1}{2}i$$

$$e' = +\tfrac{1}{2}ai$$

and

$$e'' = -\tfrac{1}{2}ai$$

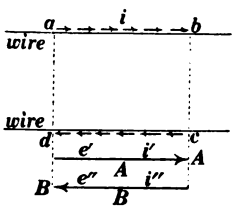

Fig. 9.

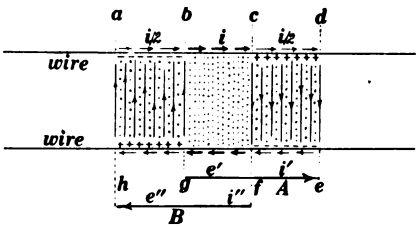

Fig. 10.

The two pure waves A and B travel towards the right and towards the left, respectively, at velocity $v = \sqrt{\frac{1}{LC}}$, and the state of things after a very short time is as shown in Fig. 10. The central region *bcfg* in Fig. 10 where the two waves A and B still overlap is in its initial condition with current i in it and zero voltage across it, the region *cdef*, where wave A exists alone has current i' in it and voltage e' across it, and the region *abgh* where wave B exists alone has current i'' in it and voltage e'' across it.

After further lapse of time the two waves A and B become wholly separated, as shown in Fig. 11, and, of course, each wave carries its voltage and current distributions with it.

The short arrows in Figs. 9 to 11 represent currents, the dots represent magnetic lines of force (perpendicular to the plane of the paper), the fine vertical lines represent electric

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lines of force, and the positive and negative signs represent electric charges.

It may be asked "How do we know that Figs. 9, 10 and 11 show what actually happens after the instantaneous removal of the bridging wires in Fig. 7?" The complete answer to this question, however unsatisfactory it may seem to be, is "Because the two pure waves A and B taken singly or together constitute solutions of the differential equations $\frac{d^2e}{dt^2} = v^2\frac{d^2e}{dx^2}$ and $\frac{d^2i}{dt^2} = v^2\frac{d^2i}{dx^2}$, and because the two waves A and B together simulate the actual physical fact at the instant of removal of the bridging wires, the fact, namely, that current i exists over the given section of the line without any voltage across the line anywhere."

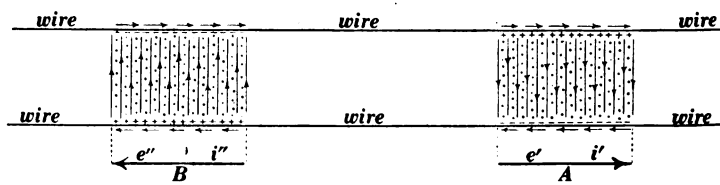

Fig. 11.

Example 2. Imagine a portion of a one-wire line (with ground return) to be charged positively by a cloud in a thunderstorm, and let us suppose that the portion of the line is uniformly charged for the sake of simplicity. The charge on the line is "bound" by the lines of force which go to the negatively charged cloud as shown in Fig. 12.

If the cloud is suddenly discharged by a lightning stroke the charge on the wire becomes "free" and the lines of force are quickly rearranged as indicated in Fig. 13 with a certain voltage e from wire to ground. This supposedly uniform voltage (with no associated current) constitutes an impure wave, and it is equivalent to the two oppositely traveling pure waves A and B. Let the student construct two diagrams somewhat similar to Figs. 10 and 11, which show the distribution of voltage and current over wire and ground in

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Fig. 13; first when the two pure waves still overlap half-and-half, and second when the two pure waves have become entirely separated.*

13. Satisfying of initial conditions. It is convenient to speak of the instant of removal of the two bridging wires in the above example 1 as the instant $t = 0$. The combination of the two pure waves A and B in Figs. 9, 10 and 11 is the solution of the problem as to what takes place on the transmission line after the removal of the bridging wires, and the fact that these two waves together give the actual state of things at $t = 0$ is usually spoken of as *the satisfying of the initial conditions.*

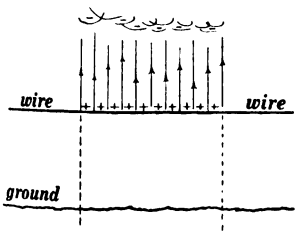

Fig. 12.

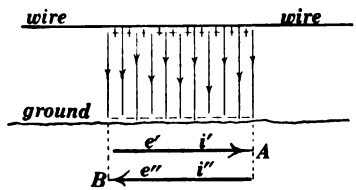

Fig. 13.

14. Satisfying of boundary conditions. The examples which are given in Art. 12 refer to an indefinitely long line, and there is no need therefore to consider what takes place at the ends of the transmission line. In general, however, it is necessary to consider what takes place at the ends of a line. At the open end of a line, for example, *there can be no current*, and at the short-circuited end of a line *there can be no voltage*. Therefore, if the solution of the problem of Figs. 7 and 8 is to be correct for the case of a finite line, the solution as represented by the two waves A and B of Figs. 9, 10 and 11 must be modified so as to meet the conditions, whatever they may be, at the ends of the line. This modification of a wave solution is usually spoken of as *the satisfying of boundary*

* Formulas for calculating the constants L and C of a one-wire line with ground return are derived in Appendix A for the ideal case in which the ground is assumed to be a perfect conductor.

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conditions. The satisfying of boundary conditions in a wave problem always reduces to a consideration of the character of the reflection of the wave or waves when they reach the boundary—or, in the case of a transmission line, when the waves reach the end or ends of the line.

15. Three simple examples which involve a consideration of end conditions (boundary conditions) but which do not necessarily involve a consideration of reflection. It is to be particularly noted that the following examples do not necessarily involve a consideration of reflection, but the solution may be set up in all three examples so as to involve reflection.

Example 1. Imagine a battery to be connected to one end

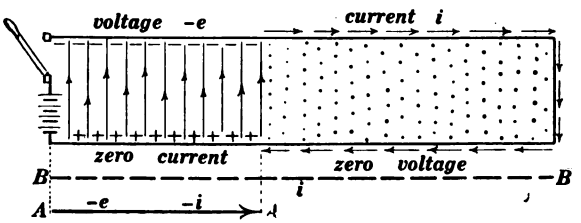

Fig. 14.

of a long line so as to establish a current i over the whole line but with zero voltage across the line everywhere, and suppose the battery switch to be opened suddenly at the home end of the line as shown in Fig. 14. What happens?

The initial uniform current i is an elementary non-wave solution of the wave equations as explained in Art. 11, and this elementary non-wave solution is symbolized by the heavy dotted line BB in Fig. 14.

The complete solution of our problem depends on our being able to devise another elementary solution of the wave equations which superposed upon BB will satisfy the necessary condition that there can be no current at the home end of the line after the battery switch is opened.

Such an additional elementary solution of the wave equations is a long-drawn-out "ribbon" wave AA which

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shoots out from the home end of the line. This "ribbon" wave consists of a uniform current $-i$ (equal and opposite to the initial current in the line) and a uniform voltage equal to $-i$ multipled by $+\sqrt{L/C}$. This voltage is negative and it is represented by $-e$ in Fig. 14.

After time t the "ribbon" wave will have spread over length vt of the line, and the ribbon wave will have wiped out the initial current and will have laid down a uniform voltage $-e$ over this portion of the line.

Example 2. An indefinitely long line which is open at the distant end is connected to a battery at the home end and uniformly charged with voltage e across it everywhere. The

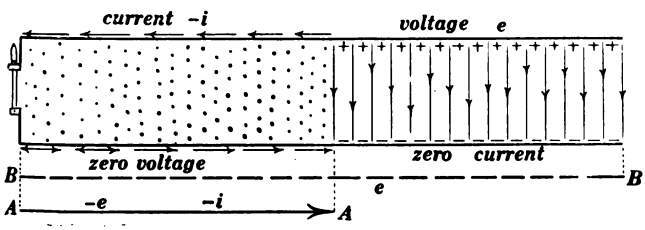

Fig. 15.

charging battery is then disconnected and the home end of the line is suddenly short-circuited. What happens?

The uniform voltage e is an elementary non-wave solution of the wave equations as explained in Art. 11, and this elementary non-wave solution is symbolized by the heavy dotted line BB in Fig. 15.

The complete solution of our problem depends on our being able to devise another elementary solution of the wave equations which superposed upon BB will satisfy the necessary condition that there can be no voltage across the home end of the line after it is short-circuited.

Such an additional elementary solution of the wave equations is the "ribbon" wave AA which shoots out from the home end of the line, wiping out the initial voltage $+e$ and laying down a uniform current $-i$ as it proceeds.

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Example 3. A double-pole switch is suddenly closed connecting a battery of voltage e (and of negligible resistance) to the home end of a long transmission line. What happens?

Initially there is neither voltage nor current distributed over the line and it is only necessary to devise a single elementary solution of the wave equations which will satisfy the necessary condition that the voltage across the home end of the line must be equal to battery voltage after the switch is closed.

Such an elementary solution of the wave equations is the "ribbon" wave AA in Fig. 16. This "ribbon" wave lays down battery voltage e and the corresponding current $i = e \times (+\sqrt{C/L})$ as it spreads over the line.

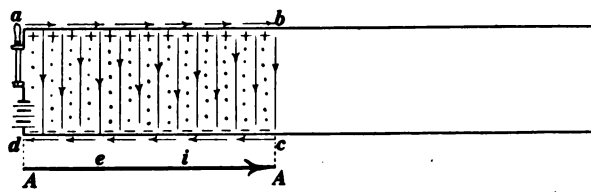

Fig. 16.

Remark. The student may be inclined to doubt the rigorous correctness of the above extremely simple solutions, but all doubts must vanish in view of the three all-important facts, namely, (1) The solutions as given are solutions of the wave equations (vii) to (x) of Art. 7; (2) The solutions as given satisfy the initial conditions; and (3) The solutions as given satisfy the boundary conditions (the end conditions).

16. Wave produced by applying a voltage e_0 to the end of a line when e_0 is any function whatever of elapsed time. A voltage e_0 which is any known function of elapsed time $[e_0 = f(t)]$ is connected across the home end of a very long line (line losses equal to zero) and it is required to find an expression for the voltage e throughout the wave which shoots out on the line, e being the voltage across the line at time t at a point on the line distant x from the home end. It is evidently necessary to express e as a function of x and t.

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When a varying voltage e_o is connected across the home end of the line *each instantaneous value of* e_o is propagated along the line at the velocity v of light (186,000 miles per second) so that the value of e at time t and at point x is the value of e_o at time $t - x/v$ because x/v is the time required for the wave to travel the distance x. Therefore if $e_o = f(t)$

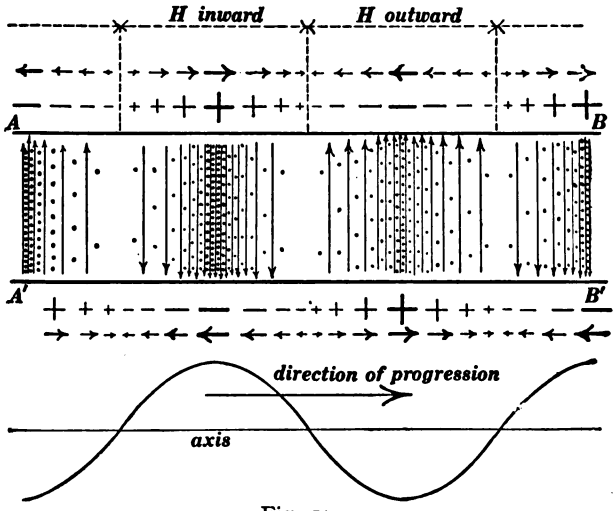

Fig. 17.

HARMONIC WAVE-TRAIN

The short horizontal arrows represent currents in the wires, the plus and minus signs represent charges on the wires, the dots represent magnetic lines of force perpendicular to the plane of the paper, and the fine vertical arrows represent electric lines of force.

it is evident that $e = f(t - x/v)$. If we know e_o as a function of t we get e as a function of x and t by putting $(t - x/v)$ for t.

Example 1. The long-drawn-out "ribbon" wave in Fig. 16 is a particular case of the above where the applied voltage e_o is constant.

Example 2. If the applied voltage e_o is a harmonic alternating voltage expressed by

$$e_o = E \sin \omega t$$

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then the expression for the voltage e in the wave is

$$e = E \sin\left[\omega\left(t - \frac{x}{v}\right)\right] \quad \text{(i)}$$

and the expression for the current i in the wave is $i = e/a$ or

$$i = \frac{E}{a} \sin\left[\omega\left(t - \frac{x}{v}\right)\right] \quad \text{(ii)}$$

Equations (i) and (ii) express what is called a *harmonic wave-train.* Such wave-trains are discussed at length in Chapter II. A complete snap-shot picture of such a wave-train is shown in Fig. 17. It must be remembered that we are here dealing with a very long line, or, at any rate, we are not concerned with complications due to reflection from the distant end of the line.

17. Wave reflection. What happens when the wave A in Fig. 11, for example, reaches the end of the line? Whatever happens, the voltage and current distributions over the line must be a solution of the wave equations (vii) to (x) of Art. 7 (must, in general, be the sum of two or more elementary solutions of the wave equations), and the end conditions must be satisfied. If we meet these two requirements the problem as to what happens at the end of the line is solved.

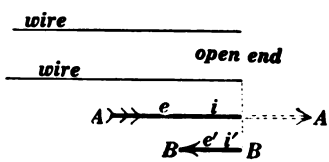

Fig. 18.

Reflection from the open end of a line. Let the arrow AA in Fig. 18 show the position in which the wave AA of Fig. 11 would be at a given instant if the wave had not reached the end of the line. What has become of the portion of the wave AA which is represented by the dotted portion of the arrow in Fig. 18? If we assume that this portion of the wave has been reflected as the wave BB in Fig. 18, and if we determine the voltage and current e' and i' so as to satisfy the end conditions, then we may be sure that the problem is solved.

The wave AA in Fig. 18 is supposed to be given, that is to

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say, e and i are supposed to be known, and, of course, the ratio e/i is equal to $+\sqrt{L/C}$ according to Art. 8.

In order that BB may be an elementary solution of the wave equations (a pure wave) the ratio e'/i' must be equal to $-\sqrt{L/C}$, according to Art. 9. Therefore we have

$$\frac{e'}{i'} = -\frac{e}{i} \quad \text{(i)}$$

Furthermore, the necessary condition at the open end of a line is *zero current*. Therefore we must have

$$i + i' = 0 \quad \text{(ii)}$$

Solving these two equations for e' and i' we get

$$e' = e \quad \text{and} \quad i' = -i$$

These results show first that the wave AA is *completely reflected* by the open end of the line because voltage and current in the reflected wave are numerically equal to voltage and current, respectively, in the original or incident wave, and second, that reflection takes place at the open end of a line with *reversal of current*.

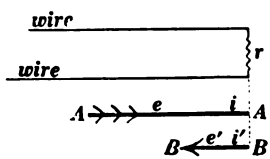

Fig. 19.

Reflection from the short-circuited end of a line. An argument very similar to the above, based on the necessary condition of *zero voltage* at the short-circuited end, shows that a wave is also *completely reflected* by the short-circuited end but with *reversal of voltage*.

18. Reflection from the end of a line across which a non-inductive circuit of resistance r is connected. Fig. 19 represents a wave AA (like the wave A of Fig. 11) partly turned back or reflected under the conditions as stated. In this case, as in Art. 17, we must have

$$\frac{e'}{i'} = -\frac{e}{i} \quad \text{(i)}$$

Now the total voltage across the end of the line is $e + e'$, the total current at the end of the line is $i + i'$, and this current is pushed through the non-inductive resistance r by the total

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voltage $e + e'$. Therefore the necessary end condition is expressed by the equation

$$e + e' = r(i + i') \quad \text{.....(ii)}$$

Solving these two equations, we get

$$e' = -e\left(\frac{e - ri}{e + ri}\right) \quad \text{.....(iii)}$$

and

$$i' = +i\left(\frac{e - ri}{e + ri}\right) \quad \text{.....(iv)}$$

or, using a for the ratio e/i which is equal to $+\sqrt{L/C}$, we get

$$e' = -e\left(\frac{a - r}{a + r}\right) \quad \text{.....(v)}$$

and

$$i' = +i\left(\frac{a - r}{a + r}\right) \quad \text{.....(vi)}$$

It is highly interesting to note that there is no reflected wave at all when $r = a$, that is to say, when $r = \sqrt{L/C}$. In such a case the incident wave in Fig. 19 is wholly absorbed or "swallowed up" when it reaches the end of the line. When non-inductive resistance a ($= \sqrt{L/C}$) is connected across the end of a line the line behaves as if it were infinitely long, that is to say, the resistance a is equivalent to an infinite extension of the line.

When r is less than a we have partial reflection (e' and i' numerically less than e and i) with reversal of voltage, and when $r = 0$ we have complete reflection with reversal of voltage.

When r is greater than a we have partial reflection with reversal of current, and when $r = \infty$ we have complete reflection with reversal of current.

19. "Discharge" of an inductive circuit into a long transmission line. Figure 20 shows an inductive circuit of resistance r and inductance l connected across the home end of a long transmission line. Let us suppose that at the beginning (when $t = 0$, where t is elapsed time) there is a known current I in rl, with no current in and no voltage across the line anywhere. The current in rl will die away and develop a

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a wave BB which will shoot out on the line. It is required to determine the wave BB.

Let e and i be the voltage across and the current in the line due to the wave BB, that is to say, e and i are the voltage and current which constitute the wave BB. Then the values of e and i at any given point on the line vary with t, and the values of e and i vary from point to point on the line at any given instant. That is to say, e and i depend on x and t, and the wave BB will be completely determined when both e and i are expressed as known functions of x and t. The short arrows e and i in Fig. 20 show the chosen positive directions of e and i at a point on the line.

Let e_o and i_o be the values of e and i at $x = 0$ (at the end O

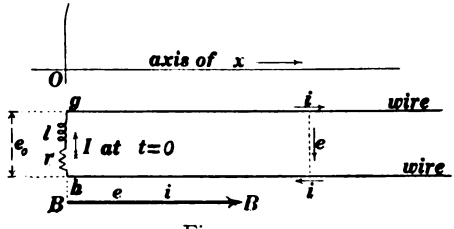

Fig. 20.

of the line). Then the upward current in rl is i_o. As this current in rl dies away the self-induced electromotive force $-l\frac{di_o}{dt}$ is partly used to overcome the resistance r, the part so used is ri_o and the remainder of the self-induced electromotive force is the voltage e_o. Therefore

$$-l\frac{di_o}{dt} = ri_o + e_o$$

But the ratio e/i in the wave BB is always and everywhere equal to $+\sqrt{L/C} = a$, according to Art. 8, and since e_o and i_o are particular values of e and i it is evident that $e_o/i_o = +a$ so that $e_o = ai_o$. Therefore the above equation reduces to

$$l\cdot\frac{di_o}{dt} + (a + r)i_o = 0 \quad \ldots\ldots\ldots\ldots\ldots \text{(i)}$$

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Remark. It was pointed out in Art. 18 that an indefinite extension of a line beyond any chosen point is exactly equivalent to a non-inductive resistance a connected across the line at that point, and the indefinitely long line in Fig. 20 is exactly equivalent to a non-inductive resistance a. That is to say, the inductive circuit rl in Fig. 20 " discharges " into the line exactly as it would discharge through a non-inductive resistance a. From this point of view equation (i) is at once evident.

Equation (i) is similar to the familiar differential equation of decay of current in an inductive circuit and its general solution is given in Appendix C, namely,

$$i_o = A\epsilon^{-\left(\frac{a+r}{l}\right)\cdot t} \quad \text{.....(ii)}$$

where A is an undetermined constant of integration. Now $i_o = I$ at $t = 0$, so that $A = I$ and equation (ii) becomes

$$i_o = I\epsilon^{-\left(\frac{a+r}{l}\right)\cdot t} \quad \text{.....(iii)}$$

and since $e_o = ai_o$ we have

$$e_o = aI\epsilon^{-\left(\frac{a+r}{l}\right)\cdot t} \quad \text{.....(iv)}$$

The voltage e_o may be thought of as the voltage which generates the wave BB in Fig. 20, and equation (iv) expresses e_o as a function of time. Therefore, according to Art. 16 we must put $(t - x/v)$ for t in equation (iv) to get an expression for e. That is to say, the exponent in equation (iv) must be $-\frac{a+r}{l}\left(t - \frac{x}{v}\right)$ or $\frac{a+r}{vl}(x - vt)$. Therefore

$$e = aI\epsilon^{\frac{a+r}{vl}(x - vt)} \quad \text{.....(v)}$$

and

$$i = I\epsilon^{\frac{a+r}{vl}(x - vt)} \quad \text{.....(vi)}$$

Neither x nor t are to have negative values assigned to them in these equations, and no value is to be assigned to x which is greater than vt.

Equation (vi) is visualized in Fig. 21 for a particular value of the time. The ordinates of the curve show the voltage values in the waves BB in Fig. 20 at a particular instant t.

20. Reflection of a long-drawn-out ribbon wave, like AA in Fig. 16, from the end of a transmission line across which a circuit

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of resistance r and inductance l is connected. Figure 22 shows the arrangement. The ribbon wave AA is taken as coming from the right so that we can conform most easily to the usual conventions as to positive directions of voltage and current. The letters e' and i' represent the voltage and current which constitute the ribbon wave AA; both e' and i' are assumed to be constant (do not vary with t) and uniform (do not vary with x); and the ratio e'/i' is equal to $-\sqrt{L/C} = -a$ according to Art. 9. The ribbon wave is supposed to be given, that is to say, e' and i' are supposed to be known. Elapsed time t in the following discussion is reckoned from the instant ($t = 0$) when the head of the ribbon wave first reaches rl.

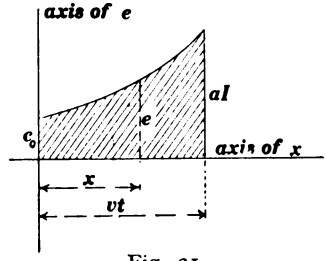

Fig. 21.

The letters e and i represent the voltage and current distributions which constitute the reflected wave BB, e and i both vary with x and both vary with t, that is to say, the voltage and current distributions e and i are neither uniform nor constant; but the ratio e/i is everywhere and always equal to $+\sqrt{L/C} = +a$ according to Art. 8. In the following discussion e_o is the value of e when $x = 0$ and i_o is the value of i when $x = 0$.

The total voltage across rl is $e_o + e'$ and the total (downward) current in rl is $-i_o - i'$. Therefore the necessary end condition is expressed by the equation

$$l \cdot \frac{d(-i_o - i')}{dt} + r(-i_o - i') = e_o + e'$$

Considering that i' is constant; and putting $-ai'$ for e' and $+ai_o$ for e_o this equation becomes

$$l \cdot \frac{di_o}{dt} + (a + r)i_o = (a - r)i' \quad \text{.....(i)}$$

and evidently $(a - r)i'$ is a constant voltage. This differential equation is therefore similar to the familiar differential equation

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of growth of current in a circuit of inductance L and resistance R when the circuit is acted on by a constant voltage. The well-known general solution of (i) as given in Appendix C is

$$i = \frac{E}{R} + A\epsilon^{-\frac{R}{L}\cdot t} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

where R* is the resistance of the circuit, L* is its inductance, E is the constant voltage which acts on the circuit, and A is an undetermined constant of integration. Therefore, using

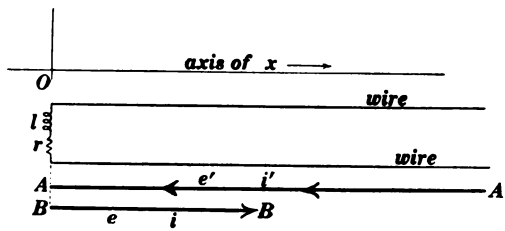

Fig. 22.

l for L, using $(a + r)$ for R, and using $(a - r)i'$ for E in equation (ii) we get

$$i_o = \left(\frac{a - r}{a + r}\right) i' + A\epsilon^{-\left(\frac{a + r}{l}\right)\cdot t} \quad \ldots\ldots\ldots \text{(iii)}$$

At the instant $t = 0$ when the head of the ribbon wave AA first reaches rl the actual current $(-i_o - i')$ in rl is zero, so that when $t = 0$ we know that $i_o = -i'$. Therefore, using this pair of values of t and i_o in equation (iii) we can determine the value of A, and, using the value of A so determined, equation (iii) becomes

$$i_o = \frac{i'}{a + r}\left[a - r - 2a\epsilon^{-\left(\frac{a + r}{l}\right)\cdot t}\right] \quad \ldots\ldots\ldots \text{(iv)}$$

and for e_o which is equal to ai we get

$$e_o = \frac{ai'}{a + r}\left[a - r - 2a\epsilon^{-\left(\frac{a + r}{l}\right)\cdot t}\right] \quad \ldots\ldots\ldots \text{(v)}$$

To get expressions for e and i it is only necessary to put $(t - x/v)$ for t in these equations, as explained in Art. 16.

* Do not confuse this R and this L with resistance per unit length of line and inductance per unit length of line.

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Therefore, since $-\frac{a+r}{l}\left(t-\frac{x}{v}\right)=\frac{a+r}{vl}(x-vt)$, we have

$$e=\frac{ai'}{a+r}\left[a-r-2a\epsilon^{\frac{a+r}{vl}(x-vt)}\right] \quad \ldots\ldots(\text{vi})$$

in which neither x nor t can have negative values, and in which no value must be assigned to x which is greater than vt.

To visualize equation (vi) let us take e' as positive and i' as negative (one or the other must be negative because $e'=-ai'$ according to Art. 9), and let us suppose that $r=\frac{1}{3}a$ so that $(a-r)/(a+r)=\frac{1}{2}$. Then equation (vi) is visualized by the curve in Fig. 23. The ordinates of this curve show the voltage values throughout the wave BB in Fig. 22 at a given instant.

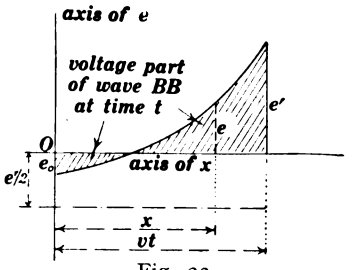

Fig. 23.

Remarks. At $t=0$ there can be no current in rl in Fig. 22, and therefore the reflection at the very beginning ($t=0$) is like the reflection that takes place at the open end of a line, that is to say, the reflection is complete and accompanied by reversal of current.

After a long time (when t is large) the current in rl settles to the constant value which is determined by Ohm's law, and therefore the final reflection from rl in Fig. 22 is like the reflection which takes place at the end of a line across which a non-inductive resistance r is connected. See Art. 18. It is evident therefore that the above equations (iv) and (v) must reduce, respectively, to equations (vi) and (v) of Art. 18 when $t=\infty$.

21. Discharge of a condenser into a long line. A condenser of capacity c with initial charge Q is connected at $t=0$ to the end of a line with a non-inductive resistance r in series as shown in Fig. 24, and it is required to determine the wave BB which shoots out on the line.

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The positive and negative signs in Fig. 24 show that the charged condenser is connected so as to produce an upward current in rc and a positive current in the line, and this line current (at the end of the line) will be at each instant equal to the rate of decrease of q where q is the charge left on the condenser at time t. Therefore

$$i_0 = -\frac{dq}{dt}$$

where i_0 is the line current at $x = 0$. Let e_0 be the line voltage at $x = 0$, and let e and i represent the voltage and current distributions which constitute the wave BB. Then e_0 and i_0 are the values of e and i at $x = 0$.

The terminal voltage of the condenser is equal to q/c and it is upwards in Fig. 24, so that this terminal voltage of the

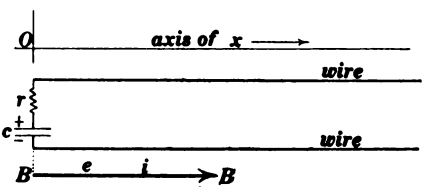

Fig. 24.

condenser is opposed by e_0 when e_0 is positive. Therefore the net upward voltage available for overcoming the resistance r in the production of the upward current i_0 is $\frac{q}{c} - e_0$ so that

$$\frac{q}{c} - e_0 = ri_0 \quad \text{.....(i)}$$

This equation expresses the end conditions.

Putting ai_0 for e_0 and putting $-\frac{dq}{dt}$ for i_0 in equation (i), we get

$$(a + r)\frac{dq}{dt} + \frac{q}{c} = 0 \quad \text{.....(ii)}$$

Remark. This equation shows that the condenser discharges into the line exactly as it would discharge through a non-inductive resistance a (in addition to r). That is to say,

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the indefinitely long line in Fig. 24 is equivalent to a non-inductive resistance a in so far as the discharging condenser is concerned.

The general solution of equation (ii), as given in Appendix C, is

$$q = A\epsilon^{-\frac{1}{(a+r)c}\cdot t} \quad \text{.....(iii)}$$

where A is an undetermined constant of integration. But when $t = 0$ we know that $q = Q$, so that $A = Q$. Using this value of A, differentiating (iii) with respect to t and remembering that $\frac{dq}{dt} = -i_o$, we get

$$i_o = \frac{Q}{(a+r)c}\cdot\epsilon^{-\frac{1}{(a+r)c}\cdot t} \quad \text{.....(iv)}$$

and, since $e_o = ai_o$, we have

$$e_o = \frac{aQ}{(a+r)c}\cdot\epsilon^{-\frac{1}{(a+r)c}\cdot t} \quad \text{.....(v)}$$

whence, using the method of Art. 16 as exemplified in Arts. 19 and 20, we get

$$e = \frac{aQ}{(a+r)c}\cdot\epsilon^{\frac{1}{(a+r)cv}(x-vt)} \quad \text{.....(vi)}$$

This equation expresses the voltage part of the wave BB in Fig. 24 and it is visualized in Fig. 25.

Remark. The problem of the reflection of a long-drawn-out ribbon wave from the end O of the line in Fig. 24 is very similar to the problem which is discussed in Art. 20, and in so far as it differs from the problem of Art. 20 it is similar to the problem which is discussed in this article. See problem 29.

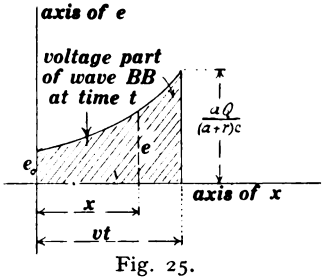

Fig. 25.

22. Wave attenuation and wave distortion.* When an electric current dies away in a simple inductive circuit (after the electromotive force of a battery, for example, has ceased

* This matter is discussed more adequately in Chapter III and the method used by the telephone engineer for calculating attenuation and distortion is outlined there.

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to act on the circuit) the magnetic energy of the circuit, $\frac{1}{2}li^2$, is converted into heat in accordance with Joule's law. Let us imagine a pure electric wave to have suddenly entered upon a transmission line of which the wire resistance is *not* negligible, then *the immediate effect of the wire resistance is to cause the magnetic energy of the wave to decrease, thus decreasing the current in the wave and causing the wave to become impure* (*ratio* e/i *no longer equal to* $\sqrt{L/C}$).

When a charged condenser is disconnected from the charging battery and left standing a very slight current flows through the imperfectly insulating dielectric, the voltage across the condenser decreases, and the electric energy of the condenser, $\frac{1}{2}ce^2$, decreases. Let us imagine a pure electric wave to have suddenly entered upon a transmission line of which leakage conductance (reciprocal of leakage resistance) is *not* negligible, then *the immediate effect of the poor insulation of the line is to cause the electric energy of the wave to decrease, thus decreasing the voltage in the wave and causing the wave to become impure.*

The decrease of current and voltage in a wave due to wire resistance and leakage is called *wave attenuation.* In general, however, the decrease of current and voltage in a *line element* is not due wholly to wire-resistance and leakage loss in that element, but some energy may be carried into or out of the element by wave action of a complicated kind. This complicated wave action is called *wave distortion* and it is briefly discussed below.

Wave distortion. Imagine a pure, rectangular, electric wave like A of Fig. 11 to have entered a transmission line having appreciable wire resistance. Such a pure wave is represented in Fig. 26; the ordinate of the shaded rectangle in Fig. 26 represents the values of e and i in the wave. After the lapse of time Δt this wave will have traveled a distance $V \cdot \Delta t$ along the line, the magnetic energy of the wave and the current i in the wave will have been decreased by the wire-resistance, and the wave will have become impure. This impure wave ($e > \sqrt{L/C} \times i$) then breaks up into two pure

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waves, one of which, containing nearly all of the energy, travels forwards as the "head" of the wave, and the other, containing a small amount of energy, shoots backwards and constitutes the "tail" of the wave as shown in Fig. 27. The action is here described as taking place after the original wave has traveled a very short distance, but, of course, the action is continuous. The head of the wave becomes impure continuously because of wire resistance, and a long-drawn-out wave of low intensity shoots backwards in the form of a "tail." The energy of the head of the wave decreases because of loss of energy in the heating of the line wires and also because

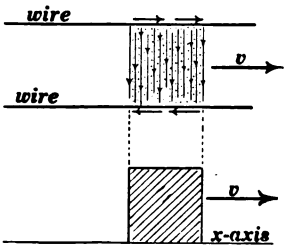

Fig. 26.

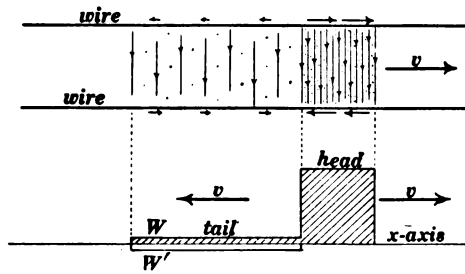

Fig. 27.

of the carrying away of energy by the long-drawn-out wave which constitutes the "tail." The head of the wave in this discussion must be supposed to cover only an infinitesimal length of the line.

Every portion of an extended wave develops a tail continuously as above described, and every portion of a tail develops its own tail. The action is evidently very complicated.

The head of a wave comes to new parts of the line continuously as the wave travels along, and the current in the line wires is almost wholly skin-current near the head of a wave. This means that the wire resistance is excessive near the head of a wave, and therefore the head of a wave is attenuated very rapidly as it travels along. This attenuation is, of course, due in part directly to the loss of energy in the

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high resistance and in part to the pronounced tail-building action which is the result of the high resistance.

Example of wave distortion due to wire resistance (skin effect ignored). Suppose that we have a line for which the value of $\sqrt{L/C}$ is one ohm, so that the pure wave in Fig. 26 may be supposed to consist of $e = 1000$ volts and $i = 1000$ amperes, and let us suppose that i is reduced to 999 amperes after a short time Δt because of the effects of wire resistance. Then the residue, 1000 volts and 999 amperes, is an impure wave which breaks up into two waves, namely, (*a*) a forward traveling wave of $999\frac{1}{2}$ volts and $999\frac{1}{2}$ amperes, and (*b*) a backward traveling wave of $+\frac{1}{2}$ volt and $-\frac{1}{2}$ ampere.

In the distortion of a narrow rectangular wave by wire resistance the voltage in the "tail" is of the same sign as the voltage in the "head" of the wave and the current in the "tail" of the wave is opposite in sign to the current in the "head" of the wave. These signs are indicated by the small arrows in Fig. 27 which shows the distorting effect of wire resistance. Also the negative ordinate of the curve W' shows the negative value of the current in the tail of the wave.

The distorting effect due to poor line insulation (leakage) is very similar to the distorting effect due to wire resistance except that in the case of poor line insulation the head of a wave becomes impure because of decrease of voltage. Thus, in the above example the head of the wave would consist of 999 volts and 1000 amperes after a short travel along the line, and this impure wave would break up into two pure waves, namely, (*a*) a forward moving wave of $999\frac{1}{2}$ volts and $999\frac{1}{2}$ amperes, and (*b*) a backward-moving wave of $-\frac{1}{2}$ volt and $+\frac{1}{2}$ ampere. The voltage in the "tail" is in this case opposite in sign to the voltage in the "head," and the current in the "tail" is of the same sign as the current in the "head."

Wave distortion is a serious matter in long-distance telephony. Thus the ordinates of the extremely irregular curve in Fig. 28 represent voltage (and current) values in a telephone

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wave, and the reproduction of the original sound in the distant telephone receiver depends on the delivery of the telephone wave at the distant end of the line *with all its fineness of detail;* but imagine what would become of the fine details of such a wave if each elementary portion of it were to develop a "tail" as explained in connection with Figs. 26 and 27! In fact, the fine details of the wave are obliterated and an unintelligible droning sound is produced in the distant telephone receiver if the telephone line is long.

23. The distortionless line. In a pure wave the magnetic energy per unit length of line is equal to the electric energy per unit length of line, as explained in Art. 10, and if the rate of loss of magnetic energy is always equal to the rate

Fig. 28.

of loss of electric energy the two kinds of energy will remain equal and the wave will remain pure. The rate of loss of magnetic energy per unit length of line is Ri^2 and the rate of loss of electric energy per unit length of line is Ge^2. Therefore, if $Ri^2 = Ge^2$ the line will be distortionless, e and i being voltage and current values in a pure wave, that is $e/i = \sqrt{L/C}$. Therefore, using L/C for e^2/i^2, the distortionless condition ($Ri^2 = Ge^2$) becomes

$$\frac{R}{G} = \frac{L}{C} \quad \text{.....(i)}$$

If a pure wave (voltage e_o and current i_o) starts out at $t = 0$ on a distortionless line, then after time t the voltage e and current i in the wave will be

$$e = e_o \epsilon^{-\frac{G}{C} \cdot t} \quad \text{.....(ii)}$$

and

$$i = i_o \epsilon^{-\frac{R}{L} \cdot t} \quad \text{.....(iii)}$$

The two exponents are equal, according to equation (i); the voltage decays with time exactly as the voltage of a charged leaky condenser decays with time, and the current decays

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with time exactly as a current in an inductive circuit dies away because of resistance.

Note that equation (i) may be written $G/C = R/L$.

24. Reduction of wire-loss of energy in a wave by inductive loading of a line. A telephone line of which the inductance is increased by the insertion of coils of wire (inductance coils) at intervals along the line as indicated in Fig. 29 is called a *loaded line.** The presence of these coils increases the resistance of the line to some extent and it also increases the capacity of the line slightly. In the following discussion this increase of resistance and capacity is ignored.

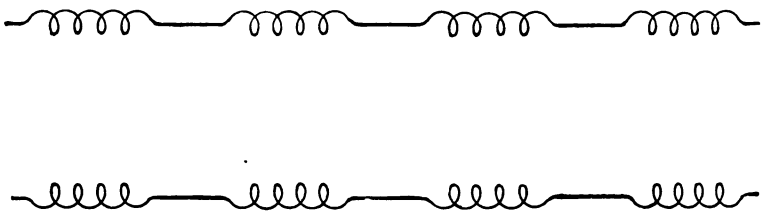

Fig. 29.

The inductive loading of a line (increase of L) decreases the value of the current for a given value of the voltage in a pure wave according to Arts. 8 and 9. A clear understanding of the effect of inductive loading may be reached by considering a particular case as follows:

Example of inductive loading of a line. Let us consider a line of such length that the time required for a wave to travel from one end of the line to the other is t seconds before the line is loaded.

1. Let us suppose that a battery of voltage e and of

* This kind of loading is called *lumpy loading* because the inductance is added in lumps. The general effect of inductive loading was first pointed out by Heaviside about 1887, and Pupin, about 1898, showed both by experiment and by a mathematical analysis that lumpy loading is equivalent to ideal smooth loading if the wave length is large as compared with the distance between successive coils in either line in Fig. 29. A line with lumpy inductance like Fig. 29 chokes out or reflects all wave lengths shorter than a certain value, that is to say, a lumpy line does not transmit such short wave-length waves to any appreciable extent. See Chapter V on Wave Filters.

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negligible resistance is connected across the home end of the line for 1/186,000 second, thus producing a rectangular wave like A in Fig. 11. This wave will be one mile long.

2. The total energy in the wave will be $\frac{1}{2}Li^2 + \frac{1}{2}Ce^2$.

3. The initial rate of loss of energy in the wires will be Ri^2.

4. The initial rate of loss of energy by leakage will be Ge^2.

These rates of energy loss fall off as the wave advances because e and i both decrease on account of attenuation. Let us assume, however, that these rates of loss remain constant; this is equivalent to assuming that the attenuation is small. Then

5. The actual wire loss of energy during the time t is Ri^2t joules.

6. The actual leakage loss of energy during the time t is Ge^2t joules.

Suppose now that L is quadrupled by loading. Then the wave velocity $v = \sqrt{1/LC}$ is halved so that the time required for a wave to travel to the distant end of the line is doubled, and if we are to produce a wave one mile long (for the sake of easy comparison) we must connect the battery for 2/186,000 second. Then

7. The battery voltage being the same as before, the current in the wave is half as large as before.

8. *The total energy of the wave is the same as before.*

9. The rate of loss of energy in the wires is one quarter as large as before.

10. The rate of loss of energy by leakage is the same as before.

11. The actual wire loss of energy during the doubled time of transit of the wave is half of what it was before.

12. The actual leakage loss of energy during the doubled time of transit of the wave is doubled.

Therefore, if the attenuation is small *the quadrupling of L by loading doubles the leakage loss of energy and halves the wire*

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loss of energy during the time of transit of a wave, the total energy of the wave being the same for the loaded and the unloaded line.

The practical importance of the inductive loading of telephone lines grows out of the fact that a well-built telephone line has very high insulation, so that the leakage loss of energy is very much smaller than the wire loss of energy, and the inductive loading of a line gives a two-fold improvement,

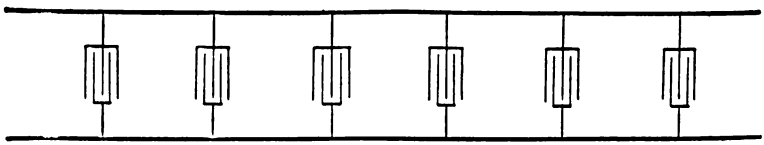

Fig. 30.

namely, (*a*) It reduces the total actual loss of energy, and (*b*) It greatly reduces wave distortion by making the wire loss of energy more nearly equal to the leakage loss of energy.

25. The capacity loading of a line. Figure 30* shows a line of which the capacity per unit length is greatly increased by connecting condensers across the line.

QUESTIONS AND PROBLEMS

Note. Except where expressly stated to the contrary all of the following problems refer to a line of zero wire resistance and zero leakage. The wires of such a line would carry skin currents only, and the inductance of such a line per unit length is given in Art. 1, Appendix A.

1. An electromotive force E acts on a non-inductive circuit of resistance r. The voltage E may be looked upon as a cause, and current and rate of development of heat as two distinct effects. Show that the currents due to two distinct voltages are superposable, and show that the rates of development of heat are not superposable.

* This figure shows a line with *lumpy capacity*. See footnote to Art. 24.

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2. A frictionless body moving at velocity v will travel a distance vt in time t. Find extra distance d the body will travel in time t if acted on by an unbalanced force F, and find the extra distance d' the body will travel in time t if acted on by an unbalanced force F', assuming initial velocity v, and the forces F and F' all to be in one fixed direction for the sake of simplicity. Considering d and d' as effects of F and F' show that these effects are superposable and explain why they are superposable.

3. If it is known that $\frac{d^2y}{dx^2}$ in Fig. 2 is constant, where y is the height of a surface or hill above the (x, z) plane, what two distinct starting curves or functions would be required in the (y, z) plane to completely determine the hill?

4. A third-order differential equation can be solved by the method of Art. 4 if the three roots of a cubic equation be found. Write down a general third-order differential equation similar to equation (i) of Art. 4, derive the cubic equation referred to, and, designating the three roots of this equation as a, b and c, write down the general solution of the third-order differential equation.

Note. The method of Art. 4 has to be modified if the determinant equation, so-called, has two or more equal roots.

5. A two-wire line consists of copper wires 0·500 inch in diameter and 24 inches apart center to center. (*a*) What is the capacity of the two-wire line per mile? (*b*) What is the inductance of the two-wire line per mile? (*c*) What is the velocity of electric wave propagation on the line? Ans. (*a*) 0·00293 henry per mile; (*b*) $9{\cdot}78 \times 10^{-9}$ farads per mile; (*c*) 186,000 miles per second.

Note. The formula $v = \sqrt{1/LC}$ gives v in miles per second when L is in henrys per mile and C is in farads per mile.

See Appendix A for derivation of formulas for L and C.

6. A 10,000-volt generator of neglible resistance and negligible inductance is suddenly connected to one end of a long line, such as is specified in problem 5. What is the voltage and what is the current in ribbon wave that shoots out on the line, what is the electric energy of each mile of the ribbon wave, and what is the magnetic energy of each mile of the ribbon wave? Ans. 10,000 volts; 18·27 amperes; 0·489 joule per mile; 0·489 joule per mile.

7. Find the diameter of each of two wires which when used as a two-wire line with a center-to-center spacing of 24 inches will

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give $\sqrt{L/C}$ equal to 500 "ohms" so that a current of 20 amperes would be associated with 10,000 volts in a pure wave. Ans. 0·745 inch.

8. A very long line with wires 0·745 inch in diameter and 24 inches apart center to center has a uniform current of 40 amperes in an 18·6-mile section near the middle of the line, with zero voltage across the line everywhere. Resolve this current distribution into two oppositely traveling pure waves and determine the actual distribution of current and voltage over the line (*a*) After 1/30,000 second, and (*b*) after 1/5000 second, neglecting wire resistance and leakage.

9. Suppose an 18·6-mile section near the middle of the line of the previous problem to have a uniform voltage of 50,000 volts across it, with zero current in the line everywhere. Resolve this voltage distribution into two oppositely traveling pure waves and determine the actual current and voltage distribution over the line (*a*) after 1/30,000 second, and (*b*) after 1/5000 second, neglecting wire resistance and leakage.

10. An 18·6-mile section near the middle of the long line of the previous problem has a uniform voltage of 30,000 volts across it (positive) and a uniform current in it of 30 amperes (positive). Resolve this current and voltage distribution into two oppositely traveling pure waves, determine the actual current and voltage distribution over the line after 1/30,000 second, and determine the actual current and voltage distribution over the line after 1/5,000 second. Neglect wire resistance and leakage.

11. A battery of which the resistance is 1,500 ohms and of which the electromotive force is 10,000 volts is suddenly connected across one end of the line of the previous problem. Find current and voltage in the ribbon wave which shoots out on the line.

12. The battery of the previous problem is suddenly connected across the line near its middle. Find current and voltage values in the ribbon waves which shoot in both directions from the battery. Ans. in ribbon wave shooting towards the right $e = 1428$ volts, $i = +28{\cdot}57$ amperes; in ribbon wave shooting towards the left $e = 1428$ volts, $i = -28{\cdot}57$ amperes.

13. A voltage $e_o = a + bt$, where a and b are constants and t is elapsed time is connected to one end of a long line like that specified in problem 7 (imagine e_o to be the electromotive force of a battery of zero resistance). Derive the equations which

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express the voltage and current distributions in the wave which is produced.

14. Imagine a battery of 300 ohms resistance whose electromotive force is $a + bt$ to be connected to one end of a long line like that specified in problem 7. Derive the equations which express the voltage and current distributions in the wave which is produced.

15. A line like that which is specified in problem 7 is 18·6 miles long. The line is open at the distant end and a 10,000-volt battery of negligible resistance is suddenly connected to the home end of the line. Determine the distribution of voltage and current over the line after 0·00015 second and after 0·00025 second. Neglect wire resistance and leakage.

16. Solve problem 15 for the case in which the distant end of the line is short-circuited.

17. Solve problem 15 for the case in which a non-inductive resistance of 250 ohms is connected across the distant end of the line.

18. Solve problem 15 for the case in which a non-inductive resistance of 1,000 ohms is connected across the distant end of the line.

19. Solve problem 15 for the case in which a non-inductive resistance of 500 ohms is connected across the distant end of the line.

20. Why should a resistance equal to $\sqrt{L/C}$ be connected in series with a spark-gap lightning arrester at the end of a transmission line?

21. Derive formulas for voltage and current in the part of a ribbon wave of voltage e and current i $(= e \times \sqrt{C/L})$ which gets past a fault on a line, the fault being a non-inductive resistance r connected across the line, also derive formulas for voltage and current in the wave which is reflected by the fault.

22. For what value of r in the previous problem will no part of the ribbon wave get past the fault? For what value of r will no part of the ribbon wave be reflected by the fault? Is there any value of r for which the ribbon wave is wholly swallowed up by the fault (no part being reflected by the fault and no part getting past the fault)?

23. The constants L and C of a long line are $4{\cdot}9 \times 10^{-3}$ henry per mile and 1×10^{-8} farad per mile respectively, the wire

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resistance of the line and the line leakage are to be neglected. A coil having an inductance l of 800 millihenrys and a resistance of 1,000 ohms is connected across the home end of the line and we will suppose that a current of 5 amperes is flowing in the coil at the instant it is connected. (*a*) What is the rate at which the current in the coil begins to decrease? (*b*) What is the time constant of the coil as connected? (*c*) What voltage and what current exist in the extreme head of the wave which shoots out from the inductance coil? (*d*) What is the voltage and current at a point 9·3 miles from the home end 0·0001 second after the inductance coil (with its 5 amperes initial-value current) is connected to the line?

24. Explain in simple physical terms why the initial reflection from the inductive circuit in Fig. 22 is exactly like the reflection from the open end of a line.

25. Explain in simple physical terms why the ultimate reflection of a long-drawn-out ribbon wave from the inductive circuit in Fig. 22 is exactly like the reflection from a non-inductive resistance r as discussed in connection with Fig. 19.

26. Explain in simple physical terms why the initial reflection from the circuit rc in Fig. 24 is exactly like the reflection from a non-inductive resistance r as discussed in connection with Fig. 19.

27. Explain in simple physical terms why the ultimate reflection of a long-drawn-out ribbon wave from the circuit rc in Fig. 24 is exactly like the reflection from the open end of a line.

28. A 10-microfarad condenser is charged by a voltage of 1,400 volts and connected in series with 1,000 ohms of non-inductive resistance to the home end of a line like that specified in problem 23. (*a*) At what rate does the voltage across the condenser begin to decrease? (*b*) What is the time constant of the condenser as connected? (*c*) What voltage and what current exist in the extreme head of the wave which shoots out from the charged condenser? (*d*) What is the voltage and current at a point 9·3 miles from the home end 0·0001 second after the charged condenser is connected to the line?

29. A long-drawn-out ribbon wave, like AA in Fig. 22, is reflected from the circuit rc in Fig. 24. Set up the equations for e and i in the reflected wave.

30. An inductance l and a resistance r are connected in series with a battery of electromotive force E, and the whole is suddenly

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connected across the home end of a long line. Neglecting the effects of wire resistance and leakage, set up equations expressing voltage and current in the wave which shoots out over the line.

31. A condenser of capacity c and a non-inductive resistance r are connected in series to a battery of electromotive force E, and the whole is suddenly connected across the home end of a long line. Find same results as in the previous problem.

32. Using the notation which is suggested by Fig. 31, formulate expressions for voltage and current E_r and I_r in the reflected wave and expressions for voltage and current E_t and I_t in the wave which enters the cable when a given wave (E and I given) on an air line comes up to the end of an underground cable, using a for $\sqrt{L/C}$ for the air line, and using b for $\sqrt{L/C}$ for the cable.

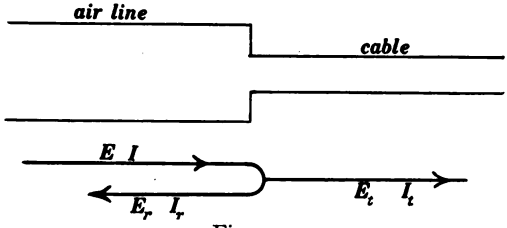

Fig. 31.

33. A transmission line A, $\sqrt{L/C} = a$, splits into two similar branch lines B, $\sqrt{L/C} = b$. A wave of given E and I on A comes up to the branch point. Derive expressions for voltage and current in the wave which is reflected back on line A, and find expressions for voltage and current in the waves which enter the two branches.

34. The three lines in the previous problem are all alike ($a = b$). Find what fraction of the energy of the incident wave on A is reflected back on A and what fraction is carried forwards in each b ranch.

Remark. The solution of the following problems 35 to 40 will give the student a clear understanding of what are usually called d.c. transients on a transmission line. The solution of these problems is only approximate because the effects of wire resistance and leakage are to be neglected. Nevertheless these approximate solutions do represent the facts with a considerable degree of accuracy for the first few thousandths of a second, and with this restriction as to time the battery of problems 35–38 may be an alternator because, whatever the value of the voltage of the

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alternator when the switch is closed, the alternator voltage will change but little in value during a few thousandths of a second.

35. A 10,000-volt battery of negligible resistance and negligible inductance is suddenly connected (at $t = 0$) to the home end of a line 93 miles long, as shown in the upper part of Fig. 32. The value of $\sqrt{L/C}$ is 500 ohms, the distant end of the line is open, and the effects of line resistance and leakage are to be ignored. Plot curves showing total voltage across the line and total current in the line as functions of elapsed time t. Plot these curves for a point x miles from the home end of the line as follows: (*a*) For

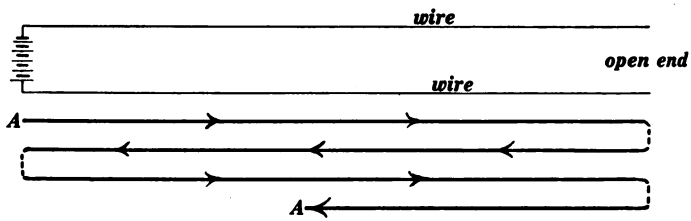

Fig. 32.

$x = 0$, (*b*) For $x = 23\frac{1}{4}$ miles, (*c*) For $x = 46\frac{1}{2}$ miles, (*d*) For $x = 69\frac{3}{4}$ miles, and (*e*) For $x = 93$ miles.

Note. A long-drawn-out ribbon wave of 10,000 volts and corresponding current shoots out from the battery and is repeatedly reflected at the ends of the line as symbolized by the long arrow AA. To solve this problem it is necessary to determine the character of the reflection at each end of the line and add voltages and currents in the successive laps of the ribbon wave.

36. Plot the curves of the previous problem for the case in which the distant end of the line is short-circuited.

37. Plot the curves of problem 35 for the case in which a non-inductive resistance of 1500 ohms is connected across the distant end of the line.

38. Plot the curves of problem 35 for the case in which a non-inductive resistance of $166\frac{2}{3}$ ohms is connected across the distant end of the line.

39. The distant end of the line in problem 35 is short-circuited and a uniform current of 20 amperes flows in the line with zero voltage across the line everywhere (this condition is thinkable because the line wires are supposed to have negligible resistance). A circuit breaker suddenly opens the circuit at the home end

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completely detaching the line from the battery. Plot the curves of problem 35.

Note. The initial uniform distribution of current is a non-wave solution of the wave equations and it may be conveniently symbolized by a heavy dotted line as in Fig. 14. This solution alone satisfies the initial conditions of the problem, and, if we assume a long-drawn-out ribbon wave of proper current value to shoot out from the opened end of the line (at the circuit breaker) and to be properly reflected back and forth at the ends of the line, the boundary conditions will be satisfied. The problem is then to be solved by adding voltages in the successive laps of the ribbon wave, and by adding currents in the laps of the ribbon wave and in the non-wave solution.

40. The transmission line of problem 35 is arranged as shown in Fig. 33. The distant end D of the line is short-circuited, a uniform current of 20 amperes flows in the line with zero voltage across the line everywhere, a circuit breaker suddenly opens the

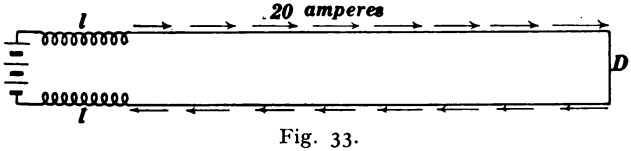

Fig. 33.

line at D, and the inductances l l are large enough to keep the current which enters the home end of the line sensibly constant for a few thousandths of a second. Plot the curves of problem 35.

41. Make a series of statements somewhat similar to the statements in Art. 24 but referring to the effect of the condenser "loading" of a line as shown in Fig. 30, taking the wire loss of energy on the unloaded line as much smaller than the leakage loss of energy.

42. A two-wire telephone line 20 miles long consists of copper wires 0·064 inch in diameter and 18 inches apart center to center. Find the line constants L, C and R and calculate the value of G for which the leakage loss of energy in a pure wave will be 2 per cent of the wire loss. Ans. $L = 0{\cdot}00408$ henry per mile; $C = 0{\cdot}00707$ microfarad per mile; $R = 26{\cdot}8$ ohms per mile; $G = 9{\cdot}30 \times 10^{-7}$ mhos (reciprocal ohms) per mile.

43. A 10,000-volt battery of negligible resistance is connected across the home end of the telephone line of the previous problem so as to produce a rectangular wave like A of Fig. 11 and 0·1 mile long. Find (*a*) total initial energy of the wave, (*b*) initial rate of energy loss in the wave due to wire resistance, (*c*) initial rate

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of loss of energy in the wave due to leakage, (*d*) total wire loss of energy during the time of transit of wave to distant end of the line on the assumption that the rate of loss continues at its initial rate, and (*e*) total leakage loss of energy during the time of transit of the wave on the assumption that the rate of loss of energy continues at its initial rate. Ans. (*a*) 0·0707 joule; (*b*) 465 watts; (*c*) 9·30 watts; (*d*) 0·050 joule; (*e*) 0·00100 joule.

44. The telephone line of the previous problem is inductively loaded so as to give a total inductance per mile 16 times as large as that of the unloaded line. The 10,000-volt battery is connected across the home end of the loaded line for sufficient time to give a rectangular wave 0·1 mile long as in problem 43. Find: (*a*) total initial energy of the wave, (*b*) initial rate of loss of energy in the wave due to wire resistance, (*c*) initial rate of loss of energy in the wave due to leakage, (*d*) total wire loss of energy during the time of transit of the wave on the assumption that the rate of loss continues at its initial rate, and (*e*) total leakage loss of energy during the time of transit of the wave on the assumption that the rate of loss continues at its initial rate. Ans. (*a*) 0·0707 joule; (*b*) 29·1 watts; (*c*) 9·7 watts; (*d*) 0·0125 joule; (*e*) 0·0040 joule.

45. Inductance is added to the telephone line of problem 43 so as to make the line distortionless, and the 10,000-volt battery is connected so as to give a rectangular wave 0·1 mile long as before. Find (*a*) Added inductance per mile, (*b*) total initial energy of the wave, (*c*) initial rate of energy loss in the wave due to wire resistance, (*d*) initial rate of energy loss in the wave due to leakage, (*e*) total wire loss of energy during time of transit of wave on the assumption that the rate of loss continues at its initial rate, and (*f*) total leakage loss of energy during time of transit of wave on the assumption that the rate of loss continues at its initial rate. Ans. (*a*) 0·200 henry per mile; (*b*) 0·0707 joule; (*c*) and (*d*) 9·3 watts; (*e*) and (*f*) 0·0107 joule.

46. Calculate the true value of (*e*) and (*f*) in the previous problem (not assuming that rate of loss continues at the initial rate). Ans. 0·00579 joule.

47. Two bare copper wires each 0.064 inch in diameter are laid 18 inches apart along the ground and the leakage resistance between wires of one mile of the two-wire line is, let us suppose, 1,000 ohms. Take values of L and C from problem 42 as applying to this line

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and find the *added* capacity per mile (see Fig. 30) to give distortionless transmission. Ans. 0·145 microfarads per mile.

48. What fraction of the initial energy of a wave would be delivered to the distant end of a 20-mile line like that specified in the previous problem when the line has sufficient capacity to make it distortionless. Ans. 0·00143 of the initial energy.

49. Transform equations (ii) and (iii) of Art. 23 so as to express e and i in terms of the distance x that the wave has traveled out from the home end of a distortionless line, e_0 and i_0 being the values of voltage and current in the wave as it starts out from the home end of the line.

50. A battery of electromotive force e and of negligible resistance is suddenly connected to the home end of a long distortionless line. Set up the equations expressing voltage and current at point x and at time t in the long-drawn-out wave which shoots out on the line.

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CHAPTER II

HARMONIC WAVES ON AN IDEAL TRANSMISSION LINE OF ZERO WIRE RESISTANCE AND ZERO LEAKAGE.*

1. Harmonic wave-train traveling towards the right in Fig. 4 of Chapter I. When a harmonic alternating voltage is applied across the home end of a long transmission line of negligible wire resistance and negligible leakage a harmonic wave-train shoots out on the line. See Example 2 of Art. 16 of Chapter I. A physical picture of such a wave-train is shown in Fig. 17 of Chapter I.

Frequency. Period. Wave-length. Amplitude. The number of complete waves that pass a given point on a line per second is called the *frequency* f of the wave-train, and the time T required for one complete wave to pass is called the *period* of the wave-train. The frequency of the wave-train is the same as the frequency of the alternating voltage which is applied to the end of the line, and, evidently $T = 1/f$.

The *wave-length* of a wave-train is the distance λ between similar points on two successive waves of the train (the distance WW in Fig. 17 of Chapter I). This is the distance traveled by the wave-train in time T, so that

$$\lambda = vT = v/f \quad \text{(i)}$$

where $v = \sqrt{1/LC}$ is the velocity of travel† of the wave-train, L being the inductance per unit length of the line, and C being the wire-to-wire capacity per unit length of the line.

* This chapter, although it deals with ideal conditions, is of great importance because it leads up to Chapter III, which deals with the actual transmission line.

† The velocity of travel of a wave train becomes a very complicated idea when wire resistance and line leakage are not negligible. See Art. 14 of Chapter III.

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The *amplitude* of a harmonic wave-train is the maximum value of the voltage, **E**, or the maximum value of the current, **I**, in the wave-train.

Equations of a harmonic wave-train traveling towards the right in Fig. 4 of Chapter I. Let

$$e_o = \mathbf{E} \sin 2\pi ft \quad \text{(ii)}$$

be a harmonic voltage acting across the home end of a long transmission line of negligible wire resistance and negligible leakage. Using the method of Art. 16 of Chapter I, it is easily shown that the expressions for voltage and current in the harmonic wave-train that shoots out on the line are

$$e = \mathbf{E} \sin \left(\frac{2\pi t}{T} - \frac{2\pi x}{\lambda}\right) \quad \text{(iii)}$$

and

$$i = \mathbf{I} \sin \left(\frac{2\pi t}{T} - \frac{2\pi x}{\lambda}\right) \quad \text{(iv)}$$

where **I** is written for $\mathbf{E}\sqrt{C/L}$. The student should derive equations (iii) and (iv). See problem 1 at the end of this chapter.

Note. The equations of a harmonic wave-train [equations (iii) to (vi)] are not used in this text except to establish a clear understanding of the clock-diagram models which are shown in Figs. 1 and 2, and these clock-diagram models are of great importance because they lead to the clock-diagram models of the next chapter.

Modification of equations (iii) and (iv) to fit Fig. 1. Equations (iii) and (iv) conform to the choice of positive directions which is shown in Fig. 4 of Chapter I, whereas it is usual in alternating current theory to measure the distance z from the "receiver end" of the line, and the "receiver end" of the line is usually shown on the right. Thus z in Fig. 1 is measured from the right towards the left, and if equations (iii) and (iv) are to fit Fig. 1 we must put $-z$ for x, so that we get

$$e = \mathbf{E} \sin \left(\frac{2\pi t}{T} + \frac{2\pi z}{\lambda}\right) \quad \text{(v)}$$

and

$$i = \mathbf{I} \sin \left(\frac{2\pi t}{T} + \frac{2\pi z}{\lambda}\right) \quad \text{(vi)}$$

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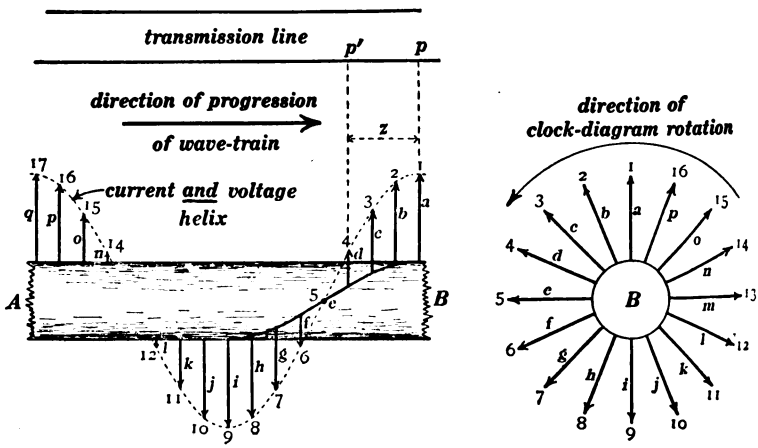
2. The clock-diagram model* of a harmonic wave-train traveling towards the right in Fig. 4 of Chapter I, z being measured as in Fig. 1 of this Chapter. This clock-diagram model is shown in Fig. 1 and the following statements are either immediately evident or easily derived from equations (v) and (vi), remembering that **I** is written for $\mathbf{E}\sqrt{C/L}$.

(*a*) The voltage e and the current i are in phase with each other at every point on the line. This means that the clock-



side view of clock-diagram model *end view of model*

Fig. 1.

diagram vector which represents e at any point on the line is coincident with the clock-diagram vector which represents i at that point.

(*b*) The ratio e/i is everywhere and always equal to $\sqrt{L/C}$, and we will represent this ratio by the letter a as in Chapter I. Therefore the voltage and current vectors are not only coincident in direction at each point, *but they may be made*

* The student is supposed to be familiar with the vector representation of alternating (harmonic) electromotive force and current. A vector diagram of this kind is called a clock diagram because the vectors are thought of as rotating.

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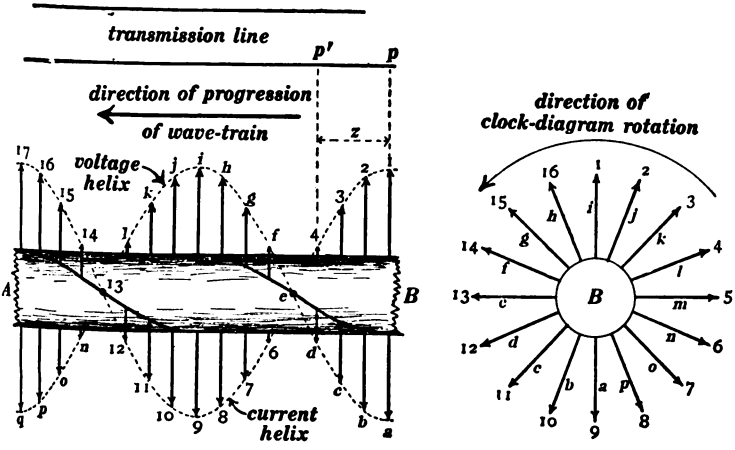
also of the same length if the voltage and current scales are properly chosen. Thus if the voltage scale is arbitrarily chosen as s volts per inch, then the current scale must be s' amperes per inch where $s' = s/a$.

(*c*) Let us choose the clock-diagram vector $a1$ in Fig. 1, the clock-diagram vector at the end of the line where $z = 0$, as our clock-diagram reference axis. Then, according to equations (v) and (vi), the clock-diagram vector $d4$ at a



side view of clock-diagram model *end view of model*

Fig. 2.

point p' on the line distant z from the end of the line (where $z = 0$) is βz radians ahead of $a1$ in phase, where $\beta z = 2\pi z/\lambda$ or $\beta = 2\pi/\lambda$.

The clock-diagram model which is shown in Fig. 1 is a number of rods set in a *left-handed* helical row in a wooden cylinder AB, all of the rods being of the same length. If the whole model is rotated f revolutions per second in a counter-clockwise direction as seen from the end B (the conventional direction of clock-diagram rotation); then the projections of the rods on the plane of the paper, as shown by the dies

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view in Fig. 1, will represent the values of e and i, projections towards the top of the page being considered as positive.

3. Clock-diagram model of a harmonic wave-train traveling towards the left in Fig. 4 of Chapter I. This clock-diagram model is shown in Fig. 2. It is a number of rods set in two *right-handed* helical rows in a wooden cylinder AB. The lettered rods in Fig. 2 represent values of the voltage e and the numbered rods (which are opposite to the lettered rods in direction) represent the values of the current i. In this wave-train (traveling towards the left) the voltage and current are everywhere opposite to each other in phase. If the whole model is rotated f revolutions per second in a counter-clockwise

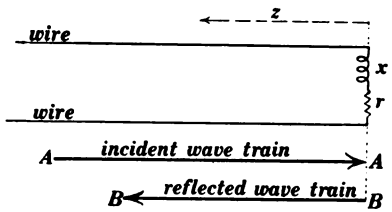

Fig. 3.

direction, as seen from the end B, then the projections of the rods on the plane of the paper, as seen in the side view, will represent the values of e and i.

4. Reflection of a harmonic wave-train from the end of a transmission line. Figure 3 represents a transmission line across the end of which a circuit of resistance r and reactance x* is connected. The arrow AA symbolizes a given *incident* harmonic wave-train, and the arrow BB symbolizes the *reflected* wave-train. The incident wave-train AA may be thought of as represented by the clock-diagram model of Fig. 1, and the reflected wave-train BB may be thought of as represented by the clock-diagram model of Fig. 2.

Let E_0' and I_0' be the clock-diagram vectors in Fig. 1 at the end $z = 0$ of the line. Then E_0' and I_0' represent voltage

* This is, of course, the value of the reactance for the particular frequency of the incident wave train AA.

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and current in the incident wave-train at the end of the line. *These vectors are coincident in direction and equal in length,* according to Art. 2; and they are given because the incident wave-train AA is supposed to be given.

Let E_o'' and I_o'' be the clock-diagram vectors in Fig. 2 at the end $z = 0$ of the line. Then E_o'' and I_o'' represent voltage and current in the reflected wave-train at the end of the line. *These vectors are opposite in direction but of the same length,* according to Arts. 2 and 3.

The reflected wave-train is then determined by Fig. 4, that is to say, the clock-diagram vectors E_o'' and I_o'' are determined by Fig. 4. This is evident when we consider that the diagram Fig. 4 satisfies all of the necessary conditions as follows:—

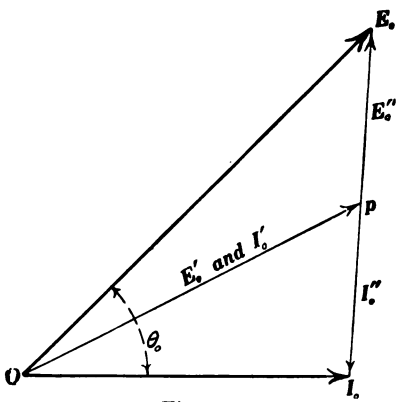

Fig. 4.

1. The clock-diagram vectors E_o' and I_o' are coincident in direction and of the same length.

2. The clock-diagram vectors E_o'' and I_o'' are opposite in direction and of the same length.

3. The total voltage E_o across the end of the line is equal to $E_o' + E_o''$.

4. The total current I_o at the end of the line is equal to $I_o' + I_o''$.

5. The ratio E_o/I_o is equal to $\sqrt{r^2 + x^2}$.

6. The angle θ_o between E_o and I_o is the angle whose tangent is x/r.

Actual determination of E_o'' and I_o'' by the graphical method, E_o', I_o', x, r and $a(= \sqrt{L/C})$ being known. Draw two lines of indefinite lengths with the angle θ_o between them, where θ_o is the angle whose tangent is x/r. These lines are OE_o and OI_o in Fig. 4. Measure off two lengths OE_o

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and OI_o whose ratio is $\sqrt{r^2 + a^2}/a$. Draw the line E_oI_o and bisect it at p. The line Op then represents E_o' and I_o' (which are given) so that by measuring the length of Op the voltage and current scales are known. Then the values of E_o'' and I_o'' can be determined from the measured lengths of the lines pE_o and pI_o, and the phases of E_o'' and I_o'' relative to E_o' are as shown in the figure.

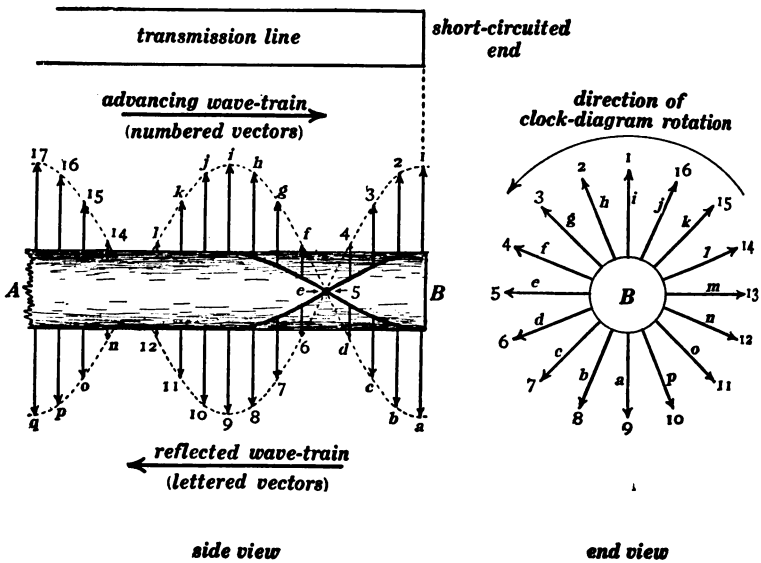

Fig. 5.

5. The standing wave-train. We have already referred to two kinds or types of solution of equations (ix) and (x) of Art. 7, Chapter I. One type of solution is described in Art. 11, Chapter I, and for lack of a better name we will call this the *dead type* of solution, because in this type of solution current and voltage are uniform and constant. Another type of solution, the *pure wave type*, is discussed in Arts. 8 and 9 of Chapter I. Still another type of solution is the *purely oscillation type* of solution, in which there is no travel in either direction. The purely oscillating type of solution of

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the differential equations of wave motion is usually called a *standing wave-train* or a *standing wave.*

Figure 5 is a clock-diagram model in which the voltage vectors from Fig. 1 and the voltage vectors from Fig. 2 are shown; the rods which stand out from the left-handed helix in Fig. 5 are the voltage vectors from Fig. 1 and their projections (as the model rotates) represent the voltage values in a wave-train traveling towards the right; the rods which stand out from the right-handed helix in Fig. 5 are the voltage vectors from Fig. 2 and their projections (as the model rotates) represent the voltage values in a wave-train traveling towards the left. The two oppositely traveling wave-trains thus represented in Fig. 5 (current vectors are omitted for the sake of clearness) are of the same wave-length and of the same amplitude, and it is both interesting and important to consider the voltage values due to the superposition of the two similar but oppositely traveling wave-trains.

Consider the two voltage vectors 1 and *a* in Fig. 5. They are equal and opposite, their resultant in zero, and there never is any voltage across the line at this point.

Consider the next pair of voltage vectors, namely, 2 and *b* in Fig. 5. They are of equal length, they stand out equally (at equal angles) from the plane of the paper in Fig. 5, and their resultant is a short vector directed towards the reader (at right angles to the plane of the paper in Fig. 5).

The next pair of voltage vectors, namely, 3 and *c*, stand out at a greater angle from the plane of the paper, and their resultant is directed towards the reader and is somewhat longer than the resultant of 2 and *b*.

Looking in this way at the successive pairs of voltage vectors in Fig. 5 one can see that their resultants all lie in a plane at right angles to the plane of the paper; these resultants are directed towards the reader from 1*a* to 9*i*, increasing in length from 1*a* to 5*e* and decreasing in length from 5*e* to 9*i*; and these resultants are directed away from the reader from 9*i* to 17*q*, increasing in length from 9*i* to 13*m*

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and decreasing in length from 13*m* to 17*q*. In fact, these resultant voltage vectors are represented by the lines *e*2, *e*3, *e*4, *e*5, etc., in Fig. 6. These lines lie in a plane and they terminate in a curve of sines in this plane, as shown in Fig. 6. See problem 8 at the end of this chapter.

If one were to bring the current vectors of Fig. 1 and 2 together in a diagram somewhat like Fig. 5, one would see that the resultants of the successive pairs of current vectors lie in the plane of the paper in Fig. 5. In fact, these resultant current vectors are represented by the lines *i*1, *i*2, *i*3, etc., in Fig. 6. They lie in a plane and they terminate in a curve of sines in this plane as shown in Fig. 6.

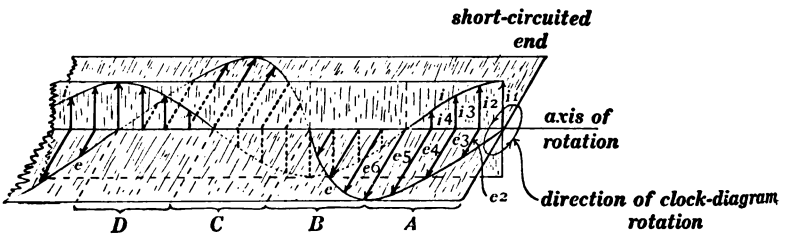

Fig. 6.

Perspective view of clock-diagram model of a standing wave-train.

Figure 6 is a clock-diagram model of a standing wave-train, and, if we imagine this model to rotate *f* revolutions per second as indicated in the figure, the projections of the voltage and current vectors on a fixed plane will represent the varying voltage and current values which exist on a transmission line when two similar oppositely traveling wave-trains are superposed, giving what is called a standing wave-train.

At certain points along the transmission line in Fig. 6 there is never any voltage. These points are called *voltage nodes*. At these points the resultant current vector is at its maximum and these points are also called *current antinodes*.

At certain points along the transmission line in Fig. 6 there never is any current, and at these points the resultant voltage

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vector is at its maximum. These points are called *current nodes* or *voltage antinodes.*

The whole distance *ABCD* in Fig. 6 is equal to one wave-length of either of the constituent traveling wave-trains (Fig. 1 or Fig. 2), and this distance is often called a wave-length of the standing wave-train. In the quarter-wave-length *A* the voltage is everywhere 90° ahead of the current in phase; in the next quarter-wave-length *B* the voltage is everywhere 90° behind the current in phase; and so on.

Another important characteristic of a standing wave-train, as represented by Fig. 6, is that at a certain instant the voltage is everywhere zero; a quarter of a cycle later the current is everywhere zero; another quarter of a cycle later the voltage is everywhere zero; and so on.

Another interesting peculiarity of the standing wave-train is that the energy shifts back and forth (is *not* transferred indefinitely as in a traveling wave-train). Thus, when the voltage is everywhere zero the energy is wholly magnetic and it is located chiefly at or near the current antinodes (the voltage nodes), whereas a quarter of a cycle later, when the current is everywhere zero, the energy is wholly electric and it is located chiefly at or near the voltage antinodes (the current nodes).

The ordinates of the two sets of sine curves in Fig. 7 represent successive current and voltage distributions along a transmission line due to a standing wave-train, the points N_i are current nodes, and the points N_e are voltage nodes.

The ordinates of the sine curve *i*1 represent currents when voltages (ordinates of curve *e*1) are everywhere zero, and the short arrows in the upper part of the figure show the actual currents at this instant.

The successive pairs of curves *i*2 and *e*2, *i*3 and *e*3, etc., show the current values as they decrease everywhere and the voltage values as they increase everywhere.

The ordinates of the sine curve *e*5 represent voltages when currents (ordinates of *i*5) are everywhere zero. At this instant

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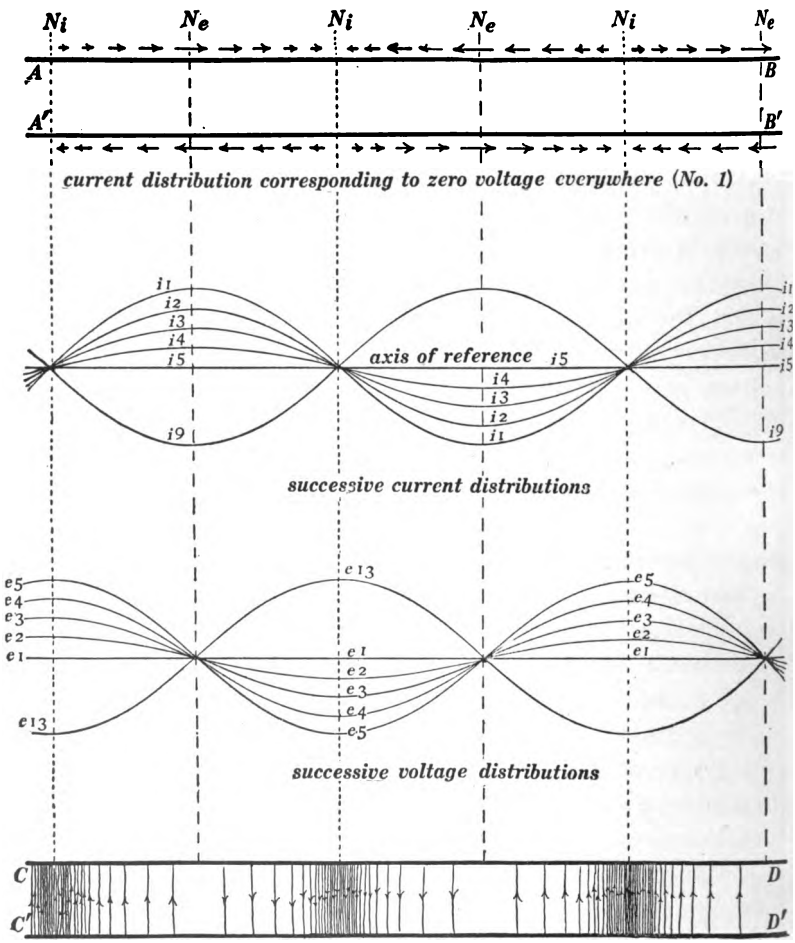
the voltages are everywhere at their highest values, and the fine vertical lines in the lower part of the figure represent the lines of force of the electric field between the wires. Where these lines of force go out from a wire the wire is positively charged, and where these lines of force come in to a wire the wire is negatively charged.



electric field (voltage) distribution corresponding to zero current everywhere (No. 5)

Fig. 7.

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Let the wave-length of a standing wave-train be λ; this is the length $ABCD$ in Fig. 6, or twice the distance between current nodes or between voltage nodes in Fig. 7. Then $\lambda = v/f$ where $v = \sqrt{1/LC}$ and f is the frequency. See equation (i) of the first article of this chapter.

Let **E** be the maximum-value of the voltage at a voltage antinode. Then the maximum-value of the voltage at a point on the line which is distant z from a voltage node is

$$\left\{\begin{matrix}\text{max.-value of voltage}\\ \text{at point distant } z\\ \text{from a voltage node}\end{matrix}\right\} = \mathbf{E} \sin \frac{2\pi z}{\lambda} \quad \ldots\ldots\text{(i)}$$

Note that 2**E** is the maximum-value of the voltage at a voltage antinode according to the answer to problem 8 at the end of this chapter, because in this problem **E** is taken as the maximum-value of the voltage in each of the traveling wave-trains which are superposed to give the standing wave-train.

Let **I** be the maximum-value of the current at a current antinode, then

$$\left\{\begin{matrix}\text{max.-value of current}\\ \text{at a point distant } z\\ \text{from a current node}\end{matrix}\right\} = \mathbf{I} \sin \frac{2\pi z}{\lambda} \quad \ldots\ldots\text{(ii)}$$

See answer to problem 8.

6. Harmonic oscillations of a transmission line of negligible wire resistance and negligible leakage. When the harmonic wave-train which shoots out from an alternator is completely reflected* at the distant end of the line, then two similar oppositely traveling wave-trains come into existence on the line and give a standing wave-train. This state of things is called a *harmonic oscillation of the line.*

It is of considerable interest to determine the voltage† amplitude of the standing wave-train (maximum-value of

* The reflection is complete, for example, when the distant end of the line is open or short-circuited, as explained in Article 17 of Chapter I.

† The current amplitude is easily determined from the voltage amplitude.

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voltage at a voltage antinode or maximum-value of current at a current antinode) for any given case, wire resistance and leakage being, of course, neglected.

Figures 8 and 9 show two typical cases of line oscillation when the distant end of the line is open, and Figs. 10 and 11

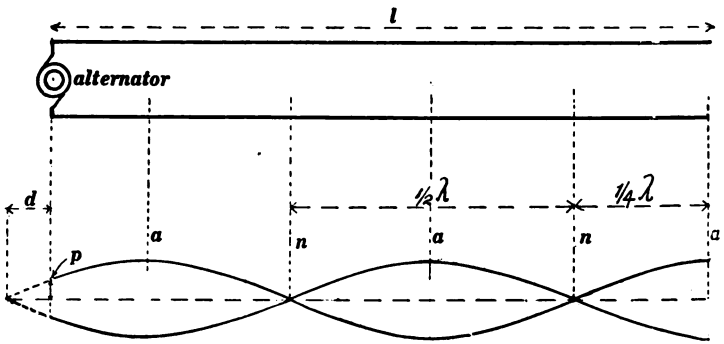

Fig. 8.

show two typical cases of line oscillation when the distant end of the line is short-circuited.

Let **G** be the maximum-value of the generator voltage and let f be the frequency. Then v/f is the wave-length of

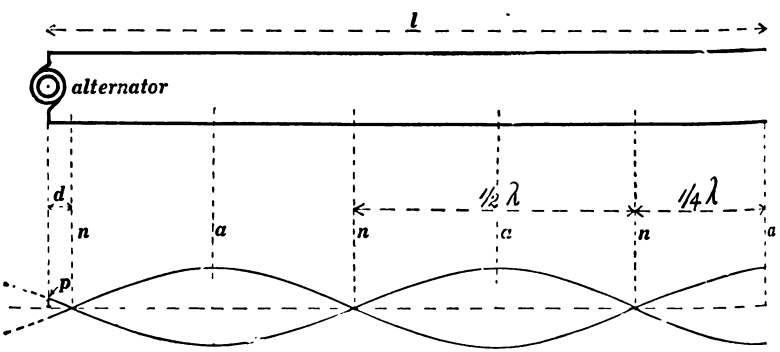

Fig. 9.

the standing wave-train where $v = \sqrt{1/LC}$ which is supposed to be given.

To determine the amplitude **E** of the standing wave-train

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in Fig. 8 or Fig. 9 calculate λ, lay off a distance $\lambda/4$ from the open end, then lay off half-wave-lengths as shown so as to locate the voltage node which is (or would be if the line were extended) at distance d from the generator where d is less

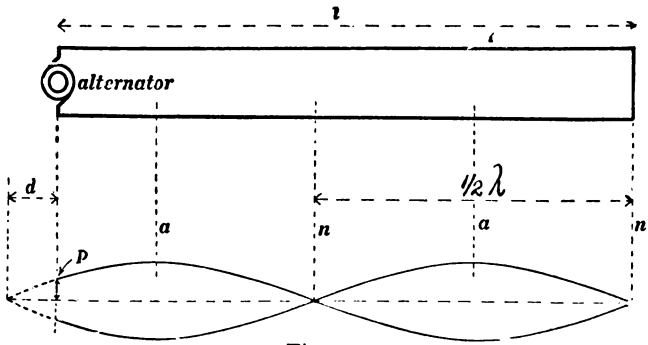

Fig. 10.

than $\lambda/4$. Then measure d, and, using d for z in equation (i) of Art. 5, we have

$$\mathbf{G} = \mathbf{E} \sin \frac{2\pi d}{\lambda} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

from which $\mathbf{E}$ can be calculated.

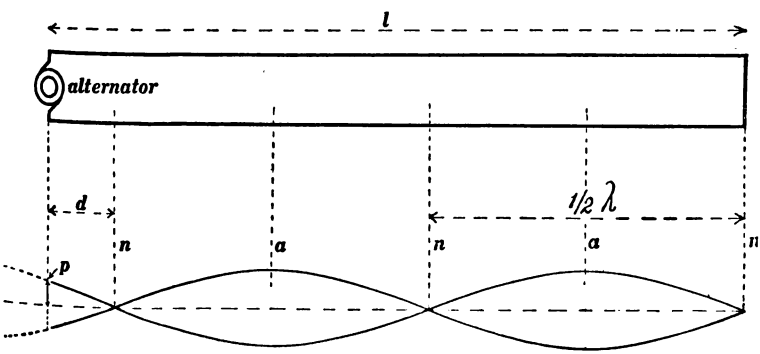

Fig. 11.

To determine $\mathbf{E}$ in Fig. 10 or 11 a somewhat similar procedure is to be followed, as will be evident at once from an inspection of the figures.

Remark. When the distance d in Figs. 8 to 11 is a very small fraction of a wave-length $\mathbf{E}$ becomes larger and larger

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as compared with **G**, and when $d = 0$ it is a physical absurdity to neglect wire resistance and leakage.

From what is said above one might be led to think that the final steady state of oscillation of the line is established when the head of the wave-train which shoots out from the generator when the generator switch is closed gets back to the generator after reflection at the distant end of the line, but this is not the case. At the instant referred to there is on the line a standing wave-train of amplitude 2**G**. The wave-train which has returned to the generator is reflected by the generator, the reflected part shoots out to the distant end of the line and back; this return wave-train is again reflected; and so on. A brief outline of the way in which the steady state is actually built up is given in Chapter III.

QUESTIONS AND PROBLEMS

1. Derive equations (iii) and (iv) of Art. 1 from equation (ii), using the method of Art. 16 of Chapter I.

2. Look at Fig. 1 which shows the position at $t = 0$ of the rotating clock-diagram model of a harmonic wave-train traveling towards the right. Take the $t = 0$ position of the current vector at end B (namely, rod No. 1) as the clock-diagram reference axis. The entire model is to be thought of as rotating f revolutions per second or $2\pi f$ radians per second in a counter-clockwise direction as seen from end B. This we will call the forward direction. Then at time t the rod No. 1 will be $2\pi ft$ radians from its $t = 0$ position in the forward direction. This angle $2\pi ft$ is called the *phase angle* of rod No. 1 at time t.

Consider the current vector which is at distance z from end B (namely, rod No. 4). This current vector is βz radians ahead of rod No. 1, so that $2\pi ft + \beta z$ is the *phase angle* of rod No. 4 at time t; where β represents $2\pi/\lambda$.

Find an expression for the upward projection i of rod No. 4 on the plane of the paper in Fig. 1, using **I** as the length of the rod,

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thus getting a general expression in terms of z and t for the current i in the wave-train which is represented by Fig. 1.

Using the same clock-diagram reference axis as before, find an expression for the upward projection e of rod d (coincident with rod No. 4), using $\mathbf{E}$ as the length of the rod; thus getting a general expression in terms of z and t for the voltage e in the wave-train which is represented by Fig. 1.

The general equations here obtained will differ from equations (v) and (vi) of Art. 1. This difference may be eliminated either by a different choice of clock-diagram reference axis (different choice of origin of time) or by a different choice of origin of z. With the above choice of origin of z what position of rod No. 1 in Fig. 1 must be chosen as the clock-diagram reference axis to give equations (v) and (vi) of Art 1? With the above choice of clock-diagram axis what origin of z will lead to equations (v) and (vi) of Art 1?

3. Let Fig. 2 be the position of the clock-diagram model at $t = 0$, take the $t = 0$ position of the current vector at end B as the clock-diagram reference axis (the numbered vectors in Fig. 2 are the current vectors), proceed as in problem 2 and set up general equations for e and i in a harmonic wave-train traveling towards the left.

4. Look at Fig. 4 of this chapter, and, interpreting this figure, state whether it is possible for the circuit rx in Fig. 3 to swallow up or absorb the incident wave-train without reflecting any; and if this is possible, state the necessary values of r and x.

5. Show from Fig. 4 that the wave-train AA in Fig. 3 is completely reflected from a pure inductance ($r = 0$), and explain the anomaly of complete reflection by a receiving circuit which takes current.

6. Show from Fig. 4 that the wave-train AA of Fig. 3 is completely reflected when rx is a condenser with lead wires of negligible resistance.

7. The line constants L and C in Fig. 3 are such that $\sqrt{L/C}$ is equal to 1,000 ohms. The incident wave-train AA has a max.-value of 10,000 volts, and the values of r and x are 300 ohms and 400 ohms, respectively, x being an inductive reactance. Find $\mathbf{I}_o'$, $\mathbf{E}_o''$ and $\mathbf{I}_o''$; find the phase angles from $\mathbf{E}_o'$ measured forwards to $\mathbf{E}_o''$ and $\mathbf{I}_o''$; and find the fraction of the energy of the incident wave that is reflected.

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8. Figure 5 shows the voltage vectors from Figs. 1 and 2 (two similar oppositely traveling wave-trains) and therefore Fig. 5 is the voltage clock-diagram model of a standing wave-train.

Let Fig. 5 represent the position of the rotating clock-diagram model at $t = 0$, and let the position of rod No. 1 at $t = 0$ be taken as the clock-diagram reference axis so that $2\pi ft$ will be the phase angle of rod No. 1 at time t.

Let z be the distance of the pair of rods 4 and d from end B of the model (see Figs. 1 and 2). Then at $t = 0$ both of these rods stand out equally from the plane of the paper in Fig. 5, making an angle of βz radians with the plane of the paper, where β is written for $2\pi/\lambda$. Write down an expression for the length in volts of the resultant of rods 4 and d, using $\mathbf{E}$ for the length of each rod.

Note that $2\pi ft + 90°$ is the phase angle at time t of the resultant of 4 and d (the forward angle from the clock-diagram reference axis to the resultant of 4 and d), and write down the expression for the voltage e in the standing wave-train.

Bringing together the current vectors from Figs. 1 and 2, and using the same clock-diagram reference axis as before, derive an expression for the current i in the standing wave-train.

Answers:

$$e = 2\mathbf{E} \sin \frac{2\pi z}{\lambda} \cos \frac{2\pi t}{\mathbf{T}}$$

$$i = 2\mathbf{I} \cos \frac{2\pi z}{\lambda} \sin \frac{2\pi t}{T}$$

Note. These answers are deliberately written down for the case in which the clock-diagram reference axis (the instant from which time is reckoned) is one quarter of a cycle earlier than is assumed in the statement of the problem, and the student is expected to check his answers by making allowance for this fact.

9. Note that equations (vii) and (viii) of Art. 7 of Chapter I refer to the choice of the x-axis which is shown in Fig. 4 of Chapter I, whereas the answers to the previous problem include z measured as shown in Figs. 1 and 2 of this chapter. Making allowance for this difference show that the answers to the previous problem satisfy equations (vii) and (viii) of Art. 7 of Chapter I and show that these answers separately satisfy equations (ix) and (x) of Art. 7, Chapter I.

10. Given a transmission line 100 miles long for which $\sqrt{L/C} = 1000$ ohms and for which $v = \sqrt{1/LC}$ is, of course, 186,000

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miles per second. An alternating-current generator whose armature has negligible resistance and negligible inductance gives an electromotive force whose maximum is 10,000 volts and whose frequency is 2,200 cycles per second is connected across the home end of the line and the distant end of the line is open. Find the positions of the voltage and current antinodes, find the maximum value of the voltage across the line at a voltage antinode, and find the maximum value of the current in the line at a current antinode. Neglect effects of wire resistance and leakage of the line. Ans. $\mathbf{E} = 24{,}150$ volts. $\mathbf{I} = 24{\cdot}15$ amperes.

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CHAPTER III

STEADY STATE OF THE ALTERNATING-CURRENT LINE

1. Notation. The theory of the steady state of the alternating-current transmission line is so complex that a carefully systematized scheme of notation is imperative. The notation used is as follows, the quantities being arranged approximately in the order in which they first appear in the text. This entire chapter refers to the simple, two-wire, single-phase line.

R = resistance of the two wires in unit length of the two-wire line.

L = inductance of unit length of the two-wire line.

$X = 2\pi fL = \omega L$ = reactance of unit length of the two-wire line.

$Z = R + jX = z/\xi$ = impedance of unit length of the two-wire line.

$z = \sqrt{R^2 + X^2}$ = numerical value of Z.

ξ = angle of Z = angle whose tangent is X/R.

G = conductance (reciprocal of insulation resistance) between wires of unit length of the two-wire line.

C = wire-to-wire capacity or capacitance of unit length of the two-wire line.

$B = 2\pi fC = \omega C$ = susceptance value of C.

$Y = G + jB = y/\zeta$ = wire-to-wire admittance of unit length of the two-wire line.

$y = \sqrt{G^2 + B^2}$ = numerical value of Y.

ζ = angle of Y = angle whose tangent is B/G.

$Q = \sqrt{ZY} = \alpha + j\beta = q/\underline{\delta}$.

$q = \sqrt{zy}$ = numerical value of Q.

$\delta = \frac{1}{2}(\xi + \zeta)$ = angle of Q.

$\alpha = q \cos \delta$.

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$\beta = q \sin \delta$.

$U = \sqrt{Z/Y} = u/\underline{\psi}$ = "surge impedance."

$u = \sqrt{z/y}$ = numerical value of U.

$\psi = \frac{1}{2}(\xi - \zeta)$ = angle of U.

l = abscissa of any point on the line measured from the receiver end.

E, E', E'', I, I', and I'' represent effective values (roots-of-mean-squares) of voltage and current at any point l on a line; prime marks refer to a decaying wave-train traveling towards the receiver end of the line, double-prime marks refer to a decaying wave-train traveling away from the receiver end of the line, and E and I refer to total actual voltage and current at the point l. Of course $E = E' + E''$ and $I = I' + I''$.

E_r, E_r', E_r'', I_r, I_r', and I_r'' are values of E, E', E'', etc., at the receiver end of the line.

Note.—When E_r is taken as the clock-diagram reference axis it is purely real.

= numerical value of I_r. See footnote in Art. 6.

= angle by which I_r leads E_r in phase; this angle is, of course, negative when I_r lags behind E_r. When E_r is taken as the reference axis the polar expression for I_r as a complex quantity is $I_r = i_r/\underline{\theta_r}$.

E_g, E_g', E_g'', I_g, I_g', and I_g'' are the values of E, E', E'', etc., at the generator end of the line.

θ_g = angle by which I_g leads E_g in phase; θ_g is, of course, negative when I_g lags behind E_g.

l' = length of fictitious line added beyond receiver end, which added line with its distant end open is exactly equivalent to the receiver. In general l' is a complex quantity.

$Z_r = E_r/I_r$ = impedance of added line l' with far end open.

E_o and I_o = voltage and current at far end of line l'; $I_o = O$ when far end of line l' is open.

2. Wire resistance and skin effect. A steady direct current spreads uniformly over the whole section of a wire, and the

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resistance which a wire offers to a steady direct current is equal to kl/a, where l is the length of the wire, a is the sectional area of the wire, and k is the specific resistance or resistivity of the material of the wire.

When a variable current flows through a wire, however, there is a tendency for the current to stay near the outside of the wire. This behavior is called *skin effect* because the flow of current is confined more or less to the parts of the wire near the surface. Skin effect exists whenever current varies, and it is due to the internal flux linkages in the conductor which are described in Art. 1 of Appendix A. The varying current causes these flux linkages to vary and eddy currents are thus set up in the wire. These eddy currents, superposed upon the primary current, produce the skin effect.

Skin effect increases the effective resistance of a wire by preventing the central parts of the wire from carrying their due share of the current, or, stated in another way, skin effect is due to eddy currents and eddy currents cause power loss. The mathematical formulation of the skin-effect problem is beyond the scope of this treatise, but the results of this formulation are set forth in the accompanying table for harmonic alternating currents.

Let r be the resistance of a cylindrical wire for a steady direct current, then sr is the resistance of the wire for harmonic alternating current, where s is a factor which depends on the size of the cylindrical conductor, on the material of which the conductor is made, and on the frequency of the alternating current flowing,* as set forth in the accompanying table.

We see from the table that skin effect is not important at ordinary frequencies until the conductors are large. The alternating-current resistance of a wire of given size comes closer and closer to direct-current resistance as the resistivity of the material of the conductor increases, and as the frequency

* See *Experimental Researches on Skin Effect in Conductors*, by Kennelly, Laws and Pierce, A.I.E.E., V. 34 (1915), p. 1953, in which a complete formulation of the skin-effect problem is made.

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of the current decreases, because these factors lower the eddy-current loss and consequently reduce the skin effect.

Skin effect is enormously increased when the conductor is made of magnetic material, because the internal flux linkages of a magnetic conductor are so high. Thus when one has a copper or aluminum cable with a steel or iron core, the skin effect in the magnetic material is so large that the steel or iron carries practically no current and the steel or iron core can be neglected entirely in computing the resistance of the cable.

TABLE

Coefficients of Skin Effect for Cylindrical Wires.

Area of conductor in circular mils.	Value of s at 60 cycles		Value of s at 25 cycles	
	Copper	Aluminum	Copper	Aluminum
2,000,000	1·302	1·132	1·065	1·025
1,750,000	1·244	1·104	1·050	1·019
1,500,000	1·188	1·072	1.037	1.014
1,250,000	1·136	1·054	1·026	1·010
1,000,000	1·091	1·035	1·017	1·006
750,000	1·053	1·020	1·009	1·003
500,000	1·024	1·009	1·004	1·002
211,600	1·004	1·002	1·001	1·000

If r is the direct-current resistance, then sr is the equivalent alternating-current resistance.

The alternating-current resistance in such a case is very nearly equal to the direct-current resistance of the non-magnetic conductor which surrounds the core.

Experiments have shown that a homogeneous stranded cable of non-magnetic material has approximately the same skin effect as a solid wire of the same material and same cross-sectional area as the stranded conductor.

3. Leakage losses of power. Idealized definition of leakance per unit length of a two-wire line. Three fairly distinct power losses are recognized as taking place on a transmission line in addition to the wire resistance loss, namely, (*a*) what may be called the *normal conduction loss* in the imperfectly insulating materials which support and surround the line wires. This is called normal conduction loss to distinguish it from corona loss

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which is also a conduction loss, (*b*) the *dielectric hysteresis loss* and (*c*) the *corona loss*.

The normal conduction loss. Because of imperfect line insulation the voltage between the line wires produces a leakage current from wire to wire, and this leakage current represents power loss. This leakage current is *not* proportional to the voltage e between the wires, that is to say, Ohm's law does *not* apply. No so-called insulating substance, with perhaps a few exceptions, conforms to Ohm's law. Doubling the voltage across the plates of a condenser does not exactly double the leakage current which is conducted through the dielectric of the condenser. The conduction loss in a cable increases with rise of temperature.

The dielectric hysteresis loss. When a condenser is charged a certain amount of work is done, but when the condenser is discharged only part of this work is given back. If the dielectric of the condenser conducts slightly some of the energy lost will be due to the leakage current. This part of the energy loss is what is described above as normal conduction loss. There is, however, some energy loss besides the normal conduction loss. This extra loss is ordinarily called *dielectric hysteresis loss*, but this term is somewhat misleading because the true dielectric hysteresis loss, which is exactly analogous to magnetic hysteresis loss in iron, is only a part and in some cases a small part of what is ordinarily called dielectric hysteresis loss.

The corona loss exists, on an air line, only when the voltage between the line wires rises above a certain critical value which depends on the pressure and temperature of the air and on the diameter and spacing of the line wires, and the corona loss of power increases very rapidly as the voltage between the line wires rises above this critical value. A very brief outline of the theory of the corona is given in Appendix E.

Idealized definition of leakance G of unit length of a two-wire line. The total leakage current between wires of a two-wire line is not proportional to the voltage between the wires but as a basis for the following development of the theory of

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the steady state of the alternating-current transmission line the total leakage current between wires of unit length of a two-wire line is assumed to be equal to Ge and the power loss due to leakage is assumed to be equal to Ge^2 where G is a constant which is, of course, expressible in reciprocal ohms. This quantity G is called the *leakance* of the line per unit length. This assumption is necessary if the mathematics of the steady state is to be feasible, and it is permissible because the leakage current and the leakage loss are usually very small; in the case of a well-constructed air line they are in fact wholly negligible when the voltage is not high enough to produce corona. That is to say, $G = o$ for a well-constructed air line.

4. Definition of the steady state of an alternating-current transmission line. When the current at every point in a transmission line is a steady alternating current and when the voltage across the line at every point is a steady alternating voltage, frequency unvarying and exactly the same everywhere, the line is said to be in the *steady state*. The following discussion of the steady state is for the case in which current and voltage are harmonic.

The steady state in acoustics. An elastic body, the air in an organ pipe, for example, can perform an infinite variety of movements; but when an organ pipe is blown steadily it settles down to a state of motion in which every particle of air in the pipe vibrates steadily at the same frequency. The amplitude of the motion varies from point to point in the pipe, but the motion at each point remains unchanged in amplitude, in wave form and in frequency. Such a state of motion of the air in an organ pipe is called a *steady state*.

Differential equations of the steady state of an alternating-current transmission line. In the whole of the following discussion current and voltage are assumed to be harmonic everywhere and the complex quantity method is used. The student who is not familiar with the complex-quantity method should read Appendix B at the end of this volume.

The differential equations of the transmission line for the steady state are derivable from the general differential equations of Art. 7, Chapter I, but it is much more instructive

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to derive the differential equations of the steady state by making use of elementary alternating-current theory as in the following discussion which is based on the notation and choice of signs which are set forth in Fig. 1.

The line current (an alternating current of frequency f) at the point p of the line distant l from the receiver end is I, the voltage across the line (an alternating voltage of frequency f) at p is E, the current in the line at p' is $I + \Delta I$, and the voltage across the line at p' is $E + \Delta E$.

The resistance, inductance, leakage conductance and capacity of the element Δl of the line are $R \cdot \Delta l$, $L \cdot \Delta l$, $G \cdot \Delta l$ and $C \cdot \Delta l$ respectively; the reactance value of $L \cdot \Delta l$ is $2\pi f L \cdot \Delta l$

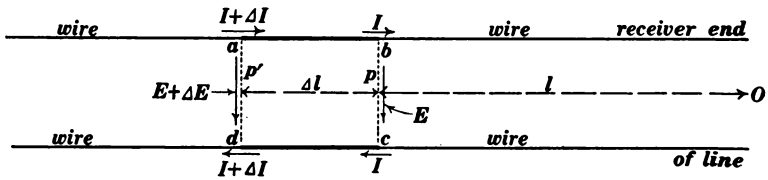

Fig. 1.

$(= X \cdot \Delta l)$ and the susceptance value of $C \cdot \Delta l$ is $2\pi f C \cdot \Delta l$ $(= B \cdot \Delta l)$.

The rise of voltage from p to p' in Fig. 1 is, of course, equal to the drop of voltage from p' to p, and this drop of voltage consists of the resistance drop $(R \cdot \Delta l)I$ and the reactance drop $j(X \cdot \Delta l)I$

so that $$\Delta E = (R \cdot \Delta l + jX \cdot \Delta l)I$$

whence $$\frac{dE}{dl} = (R + jX)I \quad \text{(i)}$$

or, using Z for $R + jX$, we have

$$\frac{dE}{dl} = ZI \quad \text{(ii)}$$

The increase of current from p to p' in Fig. 1 is, of course, equal to the decrease of current from p' to p, and this decrease of current consists of two parts, namely, the leakage current

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$(G \cdot \Delta l)E$ and the charging current $j(B \cdot \Delta l)E$. Therefore we have

$$\Delta I = (G.\Delta l + jB \cdot \Delta l)E$$

whence

$$\frac{dI}{dl} = (G + jB)E \quad \text{(iii)}$$

or, using Y for $G + jB$, we have

$$\frac{dI}{dl} = YE \quad \text{(iv)}$$

The quantity $Z(= R + jX)$ is the impedance of unit length of the line, and the quantity $Y(= G + jB)$ is the wire-to-wire admittance of unit length of the line.

To establish a differential equation involving E as the only dependent variable, equations (ii) and (iv) must be treated as simultaneous equations and I must be eliminated. To do this differentiate (ii) with respect to l and use the value of dI/dl from (iv), getting

$$\frac{d^2E}{dl^2} = ZYE \quad \text{(v)}$$

To establish a differential equation involving I as the only dependent variable, differentiate (iv) with respect to l and use the value of dE/dl from (ii), getting

$$\frac{d^2I}{dl^2} = ZYI \quad \text{(vi)}$$

It is highly desirable to express the complex quantities Z and Y in polar form, and it is convenient to represent the quantity $\sqrt{ZY}$ by the single symbol Q so that we have

$$Z = R + jX = z\underline{/\xi} \quad \text{(vii)}$$

$$Y = G + jB = y\underline{/\zeta} \quad \text{(viii)}$$

and

$$Q = \sqrt{ZY} = q\underline{/\delta} \quad \text{(ix)}$$

where

$$q = \sqrt{yz} \quad \text{(x)}$$

and

$$\delta = \tfrac{1}{2}(\xi + \zeta) \quad \text{(xi)}$$

5. Solution of equation (vi) of Art 4. Equations (v) and (vi) of Art. 4 are ordinary (not partial) differential equations, and they are exactly alike so that it is necessary to determine

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the solution of only one of them. Proceeding exactly as in Art. 4 of Chapter I, we get a solution of (vi) if we put

$$I = A\epsilon^{bl} \quad \text{(i)}$$

where A is a constant of integration and b is a constant which depends on the coefficient ZY. Indeed, following the method of Art. 4 of Chapter I, we get

$$b = \pm\sqrt{ZY} = \pm Q \quad \text{(ii)}$$

and the two values of b give us two distinct solutions of (vi), namely

$$I' = I_r'\epsilon^{Ql} \quad \text{(iii)}$$

and

$$I'' = I_r''\epsilon^{-Ql} \quad \text{(iv)}$$

We may, of course, have different values of the integration constant A in equations (iii) and (iv) and these two values of A are represented by I_r' and I_r'' which are, of course, constants. The reason for this apparently cumbersome notation will appear later.

Equations (iii) and (iv) are two distinct solutions of equation (vi) of Art. 4, and their sum is the general solution of (vi), but it is highly desirable to think of these two solutions separately in order to be able to understand them.

Meaning of the constants I_r' and I_r''. When $l = o$ it is evident that $I' = I_r'$ and that $I'' = I_r''$. Therefore I_r' is the current vector (clock-diagram vector) at the receiver end of the line which is associated with solution (iii) and I_r'' is the current vector at the receiver end of the line which is associated with the solution (iv).

Expressions for voltage. The solution of equation (v) of Art. 4 can be developed in a manner exactly similar to the above solution of equation (vi) of Art. 4, but it is better to derive the expressions for voltage from (iii) and (iv) by using equation (iv) of Art. 4 because in so doing the correct integration constants are introduced.* Therefore, differentiating (iii)

* There can be but two arbitrary integration constants in the solution of equations (ii) and (iv) of Art. 4.

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with respect to l and using equation (iv) of Art. 4, we get

$$E' = UI' \quad \text{.....(v)}$$

where

$$U = +\sqrt{Z/Y} \quad \text{.....(vi)}$$

and, differentiating (iv) with respect to l and using equation (iv) of Art. 4, we get

$$E'' = -UI'' \quad \text{.....(vii)}$$

where E' is the voltage solution which is necessarily associated with I' and E'' is the voltage solution which is necessarily associated with I''.

Decaying wave-trains. The current distribution* I' and the voltage distribution E' together constitute a *decaying wave-train* which is traveling towards the right in Fig. 1, and I'' and E'' together constitute a *decaying wave-train* which is traveling towards the left in Fig. 1. These decaying wave-trains are completely visualized by the clock-diagram models which are described in Arts. 7 and 8; but before describing these clock-diagram models it is necessary to understand the geometrical meaning of the multipliers ϵ^{Ql} and ϵ^{-Ql} and it is necessary to point out certain relations between I' and E' and between I'' and and E''.

The quantity $Q(=+\sqrt{ZY})$ is complex and to interpret the multipliers ϵ^{Ql} and ϵ^{-Ql} it is helpful to think of Q as expressed in rectangular form. Therefore we will write.

$$Q = \alpha + j\beta \quad \text{.....(viii)}$$

and the values of α and β are most easily calculated in terms of q and δ as given by equations (x) and (xi) of Art. 4, using

$$\alpha = q \cos \delta \quad \text{.....(ix)}$$

and

$$\beta = q \sin \delta \quad \text{.....(x)}$$

Substituting $\alpha + j\beta$ for Q in equations (iii) and (iv), we get

$$I' = I_r' \, \epsilon^{+(\alpha+j\beta)l} = I_r' \, \epsilon^{+\alpha l} . \, \epsilon^{+j\beta l} \quad \text{..... (xi)}$$

and

$$I'' = I_r'' \, \epsilon^{-(\alpha+j\beta)l} = I_r'' \, \epsilon^{-\alpha l} . \, \epsilon^{-j\beta l} \quad \text{..... : (xii)}$$

* I' is a function of l and E' is a function of l so that I' represents a distribution of current over the line and E' represents a distribution of voltage over the line.

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The factor $\epsilon^{+\alpha l}$ in equation (xi) *increases the magnitude* or length of the current vector I_r' and the factor $\epsilon^{+j\beta l}$ *turns* the vector I_r' *forwards* through the angle βl radians, thus converting the constant current vector I_r' at the receiver end of the line into the current vector I' at the point l of the line.

Similarly, the factor $\epsilon^{-\alpha l}$ in equation (xii) *decreases the magnitude* or length of the current vector I_r'' and the factor $\epsilon^{-j\beta l}$ *turns* the vector I_r'' *backwards* through the angle βl radians, thus converting the constant current vector I_r'' at the receiver end of the line into the current vector I'' at the point l of the line.

6. Relationship between voltage and current in a decaying wave-train. Let us represent the quantity $U = +\sqrt{Z/Y}$ in polar form thus

$$U = u\epsilon^{j\psi} = u\underline{/\psi} \quad \text{(i)}$$

then

$$-U = u\epsilon^{j(\pi+\psi)} = u\underline{/180^\circ + \psi} \quad \text{(ii)}$$

and equations (v) and (vi) of Art. 5 may be written

$$E' = (u\underline{/\psi})I' \quad \text{(iii)}$$

and

$$E'' = (u\underline{/180^\circ + \psi})I'' \quad \text{(iv)}$$

Of course, U being equal to $\sqrt{Z/Y}$, we have

$$u = \sqrt{z/y} \quad \text{(v)}$$

and

$$\psi = \tfrac{1}{2}(\xi - \zeta) \quad \text{(vi)}$$

[See equations (vii) and (viii) of Art. 4.]

From equations (iii) and (iv) it is at once evident that:

(*a*) The voltage E' in a decaying wave-train traveling towards the right in Fig. 1 is everywhere ψ radians ahead of the current I' in phase, and the ratio of the numerical values of E' and I' is everywhere $e'/i' = u$.*

(*b*) The voltage vector E'' in a decaying wave-train traveling towards the left in Fig. 1 is everywhere $180^\circ + \psi$ ahead of the

* Ordinarily e and i are used for instantaneous values of voltage and current, but it is convenient in this chapter to use small letters for the numerical values of vectors. Thus z is the numerical value of Z, y is the numerical value of Y, q is the numerical value of Q, u is the numerical value of U, etc.

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current I'' in phase, and the ratio of the numerical values of E'' and I'' is everywhere $e''/i'' = u$.

(c) Since the ratio of numerical values of E' and I' and of E'' and I'' is always equal to u it is evident that the scale of volts (s volts per inch) and the scale of amperes (s' amperes per inch) in a clock-diagram showing E' and I' or showing E'' and I'' can be so chosen as to make the length of the voltage vector the same as the length of the current vector. Let L be the length of a vector which represents E′ and suppose a vector of the

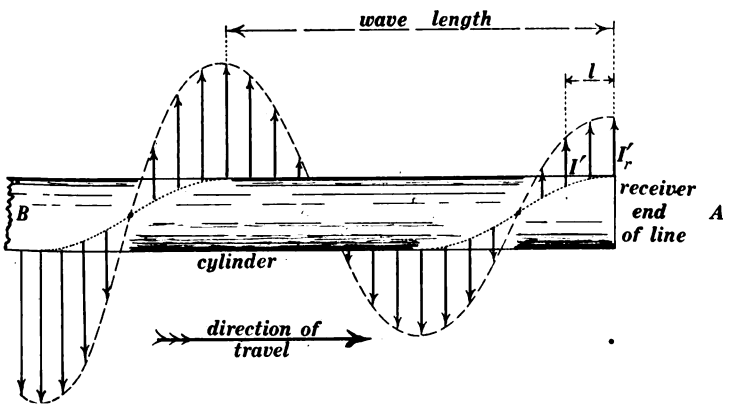

Fig. 2.

same length L represents I'. Then $E' = sL$ and $I' = s'L$ whence $E'/I' = u = s/s'$ so that $s' = us$.

7. Clock-diagram model of a decaying wave-train traveling towards the right in Fig. 1. This model is shown in Fig. 2 and it consists of rods (clock-diagram vectors) set radially in a wooden cylinder in a left-handed helical row. Rod I_r' represents the vector I_r' in equation (iii) of Art. 5 and the rod I' at any distance l from the receiver end is $\epsilon^{+\alpha l}$ times as long as I_r' and βl radians ahead of I_r' (in the conventional direction of clock-diagram rotation which is counter-clockwise as seen by an observer at A).

The rods in Fig. 2 are the current vectors, and Fig. 2 is a complete geometrical picture of equation (iii) of Art. 5. This

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equation does not contain time explicitly and neither does the model, but if we imagine the model to rotate f revolutions per second in a counter-clockwise direction as seen from A then the projections of the rods on the plane of the paper (projections considered as positive towards the top of the page) will represent the actual varying instantaneous values of current along the line.

The voltage vectors are not shown in Fig. 2 because it is better to imagine the proper positions of the voltage vectors than to attempt to show them in the figure. Now according to proposition c of Art. 6 the voltage vector at each point (each

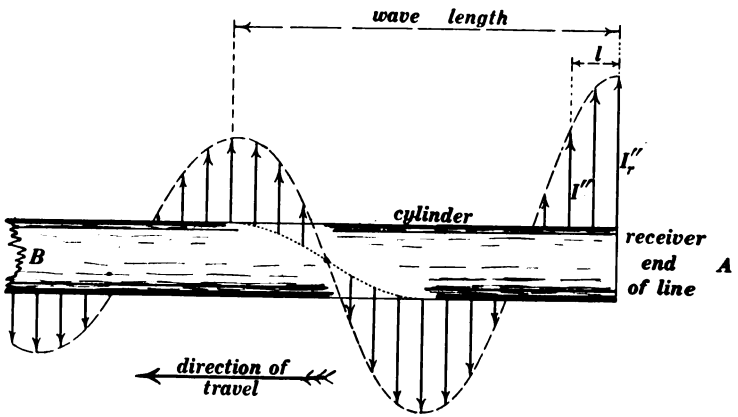

Fig. 3.

value of l) will be the same in length as the current vector at that point if $s' = us$, and according to proposition a of Art. 6 the voltage vectors are everywhere ψ radians ahead of the current vectors. Therefore voltage and current scales being properly chosen, the rods (current vectors) in Fig. 2 become voltage vectors if the whole model is turned ψ radians forwards.

8. Clock-diagram model of a decaying wave-train traveling towards the left in Fig. 1. This model is shown in Fig. 3 and it consists of rods (clock-diagram vectors) set radially in a wooden cylinder in a right-handed helical row. Rod I_r'' represents the vector I_r'' in equation (iv) of Art. 5, and the

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rod I'' at a distance l from the receiver end is shorter than I_r'' (equal to I_r'' multiplied by $\epsilon^{-\alpha l}$) and it is βl radians behind I_r''.

The voltage vectors are not shown in Fig. 3, but if the voltage and current scales are properly chosen the voltage vector at each point will be of the same length as the current vector at that point, and if the entire model is turned forwards through the angle $180° + \psi$ the rods in Fig. 3 will become voltage vectors. See equation (iv) of Art. 6.

9. Reflection of a decaying wave-train. General case. A given decaying wave-train which is symbolized by the heavy arrow AA in Fig. 4 is traveling towards the receiver end of a line, and the "receiver" has a given resistance r and a given

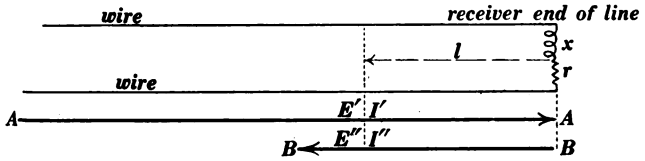

Fig. 4.

reactance x. The given wave-train AA is partially reflected by the "receiver" rx and it is desired to determine the reflected wave-train which is symbolized by the heavy arrow BB.

The given wave-train AA is represented by equations (iii) and (v) of Art. 5 and visualized in Fig. 2. The reflected wave-train BB must of necessity be represented by equations (iv) and (vii) of Art. 5 and it is visualized in Fig. 3.

The conditions which must be satisfied at the receiver end of the line are evidently

$$I_r = I_r' + I_r'' \quad \text{(i)}$$

$$E_r = E_r' + E_r'' \quad \text{(ii)}$$

and

$$E_r = (r + jx)I_r \quad \text{(iii)}$$

where $E_r'(= UI_r')$ is the voltage vector at the receiver end which is associated with the given wave-train AA, E_r'' is the voltage vector at the receiver end which is associated with the

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reflected wave-train BB, E_r, is the total voltage across the receiver end and I_r is the total current at the receiver end.

The problem is to determine E_r'' and I_r'' in terms of E_r' I_r', r, x and the line constants, and this problem can be reduced to the problem which is discussed in Art. 4 of Chapter II by turning all of the current vectors in Figs. 2 and 3 ψ radians forwards and choosing current and voltage scales in accordance with proposition c in Art. 6.

Details of the graphical solution. Let θ_r be the angle whose tangent is x/r. Draw two lines of indefinite lengths with the

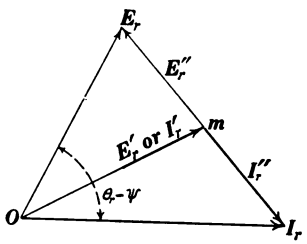

Fig. 5.

All of the current vectors are here shown ψ radians ahead of their true positions.

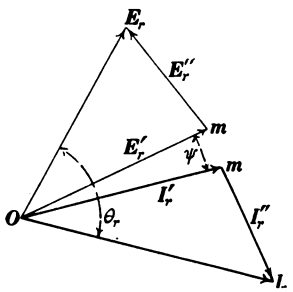

Fig. 6.

All vectors are here shown in their true positions. This is the clock-diagram at $l=o$ in Fig. 4.

angle $\theta_r - \psi$ between them. Lay off on these lines any two lengths OE_r and OI_r such that $OE_r/OI_r = \sqrt{r^2 + x^2}/u$. See Fig. 5. Bisect the line E_rI_r at m and draw Om. Then turn the triangle OmI_r in Fig. 5 backwards through the angle ψ radians thus getting Fig. 6. The line Om represents E_r'(or I_r') so that the voltage and current scales may be found by measuring Om, inasmuch as E_r' and I_r' are known. Then E_r'' and I_r'' may be determined by measuring the lines mE_r and mI_r, and the true phase relations are shown in Fig. 6.

This solution of the general case of reflection may be used to set up the equations for the two decaying wave-trains (traveling in opposite directions) which constitute the exact steady state of the

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line for any given conditions; for example when E_r, I_r and θ_r are given; **but it is very much simpler to build up the required steady state by considering two extremely simple cases of reflection as in Arts. 10 and 11.**

Establishment of the steady state on a line when a harmonic electromotive force is connected across the home end of the line. If the line is indefinitely long a "wave-train" shoots out over the line. *This initial wave-train is* **not** *a simple decaying wave-train as represented by equations (iii) and (v) of Art. 5,* but it soon settles to such a wave-train and the line is then said to be in a steady state.

If the line is not indefinitely long, then the initial "wave-train" (which is *not* a simple decaying wave-train) is partially reflected at the distant end of the line, the returning "wave-train" is partially reflected at the home end of the line, the second outgoing "wave-train" is again partially reflected at the distant end of the line, and so on until finally the state of things on the line is expressible as the sum of the two simple decaying wave-trains which are represented, respectively, by equations (iii) and (v) and by equations (iv) and (vii) of Art. 5.

10. Reflection of a decaying wave-train from the open end of a transmission line. The heavy arrow AA in Fig. 7 symbolizes a given decaying wave-train which is traveling towards the receiver end of the line. The end of the line is open and the heavy arrow BB symbolizes the reflected wave-train.

The necessary condition at the open end of the line is zero current, and therefore we must have

$$I_r' + I_r'' = o$$

This equation means, of course, that the current vectors I_r' and I_r'' are equal and opposite, and therefore, according to equations (v) and (vii) of Art. 5, E_r' and E_r'' are equal in magnitude and alike in phase.

The reflection at the open end is complete because numerically, $I_r' = I_r''$ and $E_r' = E_r''$, and the reflection takes place

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with reversal of phase of current ($I_r'' = -I_r'$) but without change of phase of voltage (E_r'' and E_r' alike in phase).

Expressions for total voltage E_p and total current I_p at any point on the line in Fig. 7 in terms of the total voltage E_r across the open receiver end. Let the subscript p be associated with the letter p in the word *open* so that the notation may be easily kept in mind.

The two wave-trains in Fig. 7 have equal and in-phase voltages at the receiver end as explained above, and the sum of these two voltages is E_r. Therefore $E_r' = E_r'' = E_r/2$ and from equations (v) and (vii) of Art. 5 we see that $I_r' = E_r/2U$

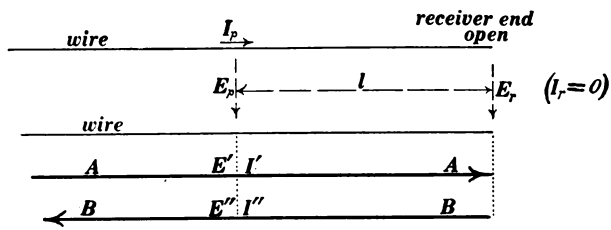

Fig. 7.

and that $I_r'' = -E_r/2U$. Consequently the analytical expressions for the two wave-trains AA and BB in Fig. 7 are

$$E' = \tfrac{1}{2}E_r\epsilon^{Ql} \quad \text{and} \quad I' = \frac{1}{2U}\cdot E_r\epsilon^{Ql} \quad \ldots\ldots\ldots\ldots \text{(i)}$$

and

$$E'' = \tfrac{1}{2}E_r\epsilon^{-Ql} \quad \text{and} \quad I'' = -\frac{1}{2U}\cdot E_r\epsilon^{-Ql} \quad \ldots\ldots\ldots \text{(ii)}$$

and, of course, the total voltage at the point l (the point on the line distant l from the receiver end) and the total current at the point l are

$$E_p = E' + E'' = E_r\left(\frac{\epsilon^{Ql} + \epsilon^{-Ql}}{2}\right) \ldots\ldots\ldots\ldots \text{(iii)}$$

and

$$I_p = I' + I'' = \frac{E_r}{U}\left(\frac{\epsilon^{Ql} - \epsilon^{-Ql}}{2}\right) \ldots\ldots\ldots\ldots \text{(iv)}$$

Remark. The expressions in the parentheses in equations (iii) and (iv) are called the *hyperbolic cosine* of Ql and the *hyperbolic sine* of Ql respectively, as stated in Art. 15, and it

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is important to note that the hyperbolic cosine and the hyperbolic sine come into the steady state solution of the transmission line because of complete reflection with reversal of current at the open end of the line, and because of complete reflection with reversal of voltage at the short-circuited end of the line as explained in the next article.

11. Reflection of a decaying wave-train from the short-circuited end of a line. The heavy arrow AA in Fig. 8 symbolizes a decaying wave-train which is traveling towards the short-circuited receiver end of a line, and the heavy arrow BB symbolizes the reflected wave-train.

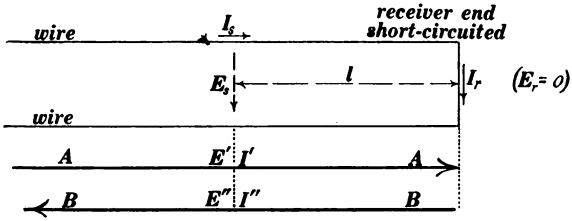

Fig. 8.

The necessary condition at the short-circuited end is zero voltage, and therefore we must have

$$E_r' + E_r'' = o$$

This equation means, of course, that E_r' and E_r'' are equal in magnitude and opposite in phase, and therefore, according to equations (v) and (vii) of Art. 5, I_r' and I_r'' are equal in magnitude and alike in phase.

The reflection at the short-circuited end is complete because, numerically, $E_r' = E_r''$ and $I_r' = I_r''$, and the reflection takes place with reversal of phase of voltage ($E_r'' = -E_r'$) but without change of phase of current (I''_r and I'_r alike in phase).

Expressions for total voltage E_s and total current I_s at any point on the line in Fig. 8 in terms of the total current I_r at the short-circuited receiver end. Let the subscript s be associated with the letter s in the word *short-circuited* so that the notation may be easily kept in mind.

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The two wave-trains in Fig. 8 have equal and in-phase currents at the receiver end as above explained, and the sum of these two currents is I_r. Therefore $I_r' = I_r'' = I_r/2$ and from equations (v) and (vii) of Art. 5 we see that $E_r' = UI_r/2$ and that $E_r'' = -UI_r/2$. Consequently the analytical expressions for the two wave-trains AA and BB in Fig. 8 are

$$E' = \tfrac{1}{2}UI_r\epsilon^{Ql} \quad \text{and} \quad I' = \tfrac{1}{2}I_r\epsilon^{Ql} \quad \ldots\ldots\ldots\ldots\text{(i)}$$

and

$$E'' = -\tfrac{1}{2}UI_r\epsilon^{-Ql} \quad \text{and} \quad I'' = \tfrac{1}{2}I_r\epsilon^{-Ql} \quad \ldots\ldots\ldots\text{(ii)}$$

and of course the total voltage at the point l and the total current at the point l are

$$E_s = E' + E'' = UI_r\left(\frac{\epsilon^{Ql} - \epsilon^{-Ql}}{2}\right) \ldots\ldots\ldots\ldots\text{(iii)}$$

and

$$I_s = I' + I'' = I_r\left(\frac{\epsilon^{Ql} + \epsilon^{-Ql}}{2}\right) \ldots\ldots\ldots\ldots\text{(iv)}$$

Remark. The symbols E', E'', I' and I'' refer to entirely different wave-trains in equations (i) and (ii) of this article and in equations (i) and (ii) of Art. 10.

12. Blondel's theorem. The actual total voltage and current at any point on a transmission line when the receiver voltage is E_r and the receiver current is I_r (leading E_r by the phase angle θ_r) is found by superposing E_p and I_p upon E_s and I_s; that is to say, by adding E_p and E_s to get the actual total voltage E and by adding I_p and I_s to get the actual total current I at any point on the line.

This theorem is a particular example of the principle of superposition, and it is true because (1) the differential equations of the transmission line are linear, and because (2) the receiver end conditions, namely, given voltage E_r and given current I_r, are satisfied by the sum of the two particular p and s solutions.

13. Expressions for total voltage E and total current I at any point l on a transmission line in terms of given E_r, I_r and θ_r at the receiver end of the line. From Blondel's theorem we have

$$E = E_p + E_s \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

and

$$I = I_p + I_s \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

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and, using the values of E_p, I_p, E_s and I_s from Arts. 10 and 11, we get

$$E = E_r\left(\frac{\epsilon^{+Ql} + \epsilon^{-Ql}}{2}\right) + UI_r\left(\frac{\epsilon^{+Ql} - \epsilon^{-Ql}}{2}\right) \ldots\ldots \text{(iii)}$$

and

$$I = I_r\left(\frac{\epsilon^{+Ql} + \epsilon^{-Ql}}{2}\right) + \frac{E_r}{U}\left(\frac{\epsilon^{+Ql} - \epsilon^{-Ql}}{2}\right) \ldots\ldots \text{(iv)}$$

These equations (iii) and (iv) constitute the complete solution of the steady state of the transmission line inasmuch as they give the values of E and I at any point on the line (any given value of l) in terms of given voltage E_r and given current I_r at the receiver end of the line.

Remark. It is to be noted that the phase difference between E_r and I_r does not enter explicitly into any of the above equations; but if E_r is taken as the clock-diagram reference axis then E_r will be purely real, and if I_r is θ_r ahead of E_r in phase then I_r will be a complex quantity whose polar expression is

$$I_r = i_r\underline{/\theta_r} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(v)}$$

14. Rearrangement of equations (iii) and (iv) of Art. 13 to facilitate computations when ordinary tables* are to be used. By regrouping the terms in equation (iii) of Art. 13 we get

$$E = \left(\frac{E_r + UI_r}{2}\right)\epsilon^{+Ql} + \left(\frac{E_r - UI_r}{2}\right)\epsilon^{-Ql} \ldots\ldots\ldots \text{(i)}$$

Using E_r as the reference axis then E_r is purely real, and the polar expressions for the remaining complex quantities are

$$U = u\underline{/\psi}$$
$$I_r = i_r\underline{/\theta_r}$$
$$\epsilon^{+Ql} = \epsilon^{\alpha l}\underline{/\beta l}$$

and

$$\epsilon^{-Ql} = \epsilon^{-\alpha l}\underline{/-\beta l}$$

where $Q = \alpha + j\beta$; and, introducing these polar forms into (i), we get

$$E = \tfrac{1}{2}\left\{E_r\underline{/\beta l} + ui_r\underline{/\beta l + \psi + \theta_r}\right\}\epsilon^{\alpha l}$$
$$+\tfrac{1}{2}\left\{E_r\underline{/-\beta l} - ui_r\underline{/-\beta l + \psi + \theta_r}\right\}\epsilon^{-\alpha l} \ldots\ldots \text{(ii)}$$

* Ordinary trigonometric and logarithmic tables.

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from which, if one is familiar with complex quantity methods, it is very easy to write out expressions for the components of E as follows

$$\left\{\begin{matrix}\text{Comp of } E\\ \text{parallel to } E_r\end{matrix}\right\} = \tfrac{1}{2}\left\{E_r \cos\beta l + ui_r \cos(\beta l + \psi + \theta_r)\right\}\epsilon^{al}$$
$$+ \tfrac{1}{2}\left\{E_r \cos\beta l - ui_r \cos(-\beta l + \psi + \theta_r)\right\}\epsilon^{-al} \ldots \text{(iii)}$$

$$\left\{\begin{matrix}\text{Comp. of } E\\ 90^\circ \text{ ahead of } E_r\end{matrix}\right\} = \tfrac{1}{2}\left\{E_r \sin\beta l + ui_r \sin(\beta l + \psi + \theta_r)\right\}\epsilon^{al}$$
$$- \tfrac{1}{2}\left\{E_r \sin\beta l + ui_r \sin(-\beta l + \psi + \theta_r)\right\}\epsilon^{-al} \ldots \text{(iv)}$$

Similarly, regrouping the terms in equation (iv) of Art. 13 we get

$$I = \tfrac{1}{2}\left(I_r + \frac{E_r}{U}\right)\epsilon^{+Ql} + \tfrac{1}{2}\left(I_r - \frac{E_r}{U}\right)\epsilon^{-Ql} \ldots\ldots\ldots\ldots \text{(v)}$$

and by using the above polar forms this becomes

$$I = \tfrac{1}{2}\left\{i_r\underline{/\theta_r + \beta l} + \frac{E_r}{u}\underline{/-\psi + \beta l}\right\}\epsilon^{al}$$
$$+ \tfrac{1}{2}\left\{i_r\underline{/\theta_r - \beta l} - \frac{E_r}{u}\underline{/-\psi - \beta l}\right\}\epsilon^{al} \ldots\ldots\ldots\ldots \text{(vi)}$$

and the expression for the components of I are

$$\left\{\begin{matrix}\text{Comp. of } I\\ \text{parallel to } E_r\end{matrix}\right\} = \tfrac{1}{2}\left\{i_r \cos(\theta_r + \beta l) + \frac{E_r}{u}\cos(-\psi + \beta l)\right\}\epsilon^{al}$$
$$+ \tfrac{1}{2}\left\{i_r \cos(\theta_r - \beta l) - \frac{E_r}{u}\cos(\psi + \beta l)\right\}\epsilon^{al} \ldots\ldots \text{(vii)}$$

$$\left\{\begin{matrix}\text{Comp. of } I\\ 90^\circ \text{ ahead of } E_r\end{matrix}\right\} = \tfrac{1}{2}\left\{i_r \sin(\theta_r + \beta l) + \frac{E_r}{u}\sin(-\psi + \beta l)\right\}\epsilon^{al}$$
$$+ \tfrac{1}{2}\left\{i_r \sin(\theta_r - \beta l) + \frac{E_r}{u}\sin(\psi + \beta l)\right\}\epsilon^{-al} \ldots\ldots \text{(viii)}$$

Note. The receiver voltage E_r is conveniently taken as the reference axis so that E_r is purely real and one might therefore use e_r for E_r. See footnote to Art. 6.

Equations (iii), (iv), (vii) and (viii) are the simplest equations obtainable for calculating voltage and current values at a point on a transmission line without the use of special tables; only ordinary trigonometric and logarithmic tables are required.

The values of ϵ^{al} and ϵ^{-al} are most easily determined by

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using a table of natural logarithms. If $k = \epsilon^{\alpha l}$ then, by definition, $\log_e k = \alpha l$ so that, αl being known, $\epsilon^{\alpha l}$ can be found by looking up the number whose natural log is αl; and, of course, $\epsilon^{-\alpha l}$ is the reciprocal of $\epsilon^{+\alpha l}$.

To carry out the calculations one must have given the values of l, E_r, i_r and θ_r and one must calculate the values of α, β, u and ψ from the known values of the line constants (R, L, G and B).

15. The use of special tables and charts to facilitate transmission line calculations. Equations (iii) and (iv) of Art. 13, which constitute the complete solution of the steady state problem of the transmission line, contain the factors

$$\frac{\epsilon^{+Ql} + \epsilon^{-Ql}}{2} \quad \text{and} \quad \frac{\epsilon^{+Ql} - \epsilon^{-Ql}}{2}$$

and these factors have been given special names. The first is called the *hyperbolic cosine* of Ql and the second is called the *hyperbolic sine* of Ql. That is to say

$$\cosh Ql = \frac{\epsilon^{+Ql} + \epsilon^{-Ql}}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

and

$$\sinh Ql = \frac{\epsilon^{+Ql} - \epsilon^{-Ql}}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

and the quantity Ql is in this connection frequently called a *hyperbolic angle.*

Using equations (i) and (ii) of this article, equations (iii) and (iv) of Art. 13 become

$$E = E_r \cosh Ql + UI_r \sinh Ql \quad \ldots\ldots\ldots\ldots\ldots \text{(iii)}$$

and

$$I = I_r \cosh Ql + \frac{E_r}{U} \cdot \sinh Ql \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(iv)}$$

When making calculations on the basis of these equations it is best to use charts which give the complex values of $\cosh Ql$ and $\sinh Ql$ in polar form as explained in Appendix D.

Let $\underline{c/c'}$ be the polar expression for $\cosh Ql$ and let $\underline{s/s'}$ be the polar expression for $\sinh Ql$. Then equations (iii) and (iv) become

$$E = E_r \underline{c/c'} + ui_r \underline{s/\psi + \theta_r + s'} \quad \ldots\ldots\ldots\ldots\ldots \text{(v)}$$

and

$$I = i_r \underline{c/c' + \theta_r} + \frac{E_r}{u} \underline{s/-\psi + s'} \quad \ldots\ldots\ldots\ldots \text{(vi)}$$

where E_r is taken as the reference axis so that E_r is purely real.

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It is very easy, if one is familiar with complex quantity notation, to write out ordinary algebraic-trigonometric expressions for the components of E and I from equations (v) and (vi). Thus

$$\left\{\begin{matrix}\text{Comp. of } E \\ \text{parallel to } E_r\end{matrix}\right\} = E_r c \cos c' + u i_r s \cos(\psi + \theta_r + s') \quad \text{(vii)}$$

$$\left\{\begin{matrix}\text{Comp. of } E \\ 90^\circ \text{ ahead of } E_r\end{matrix}\right\} = E_r c \sin c' + u i_r s \sin(\psi + \theta_r + s') \quad \text{(viii)}$$

$$\left\{\begin{matrix}\text{Comp. of } I \\ \text{parallel to } E_r\end{matrix}\right\} = i_r c \cos(c' + \theta_r) + \frac{E_r s}{u} \cdot \cos(-\psi + s') \quad \text{(ix)}$$

$$\left\{\begin{matrix}\text{Comp. of } I \\ 90^\circ \text{ ahead of } E_r\end{matrix}\right\} = i_r c \sin(c' + \theta_r) + \frac{E_r s}{u} \cdot \sin(-\psi + s') \quad \text{(x)}$$

To carry out calculations by means of these equations one must have given the quantities E_r, i_r and θ_r; one must know the distance l from the receiver end of the line to the point for which the voltage and current are to be calculated; one must calculate the quantities u, ψ and Ql from the known values of the line constants R, L, G and B; and one must find the polar expressions for sinh Ql and cosh Ql from the Kennelly charts as explained in Appendix D.

Note. Numerical calculations using equations (vii) to (x), above, and using the Kennelly charts as explained are briefer than numerical calculations using equations (iii), (iv), (vii) and (viii) of Art. 14. and using ordinary trigonometric and logarithmic tables (or a slide rule). See notes to problems 4 and 5 at the end of this chapter.

16. Abbreviated formulas for calculating E and I at many points along a line. Given the values of E_r and I_r (E_r may be taken as purely real and $I_r = i_r\underline{/\theta_r}$) at the receiver end of a line, to calculate E and I at any point on the line distant l from the receiver end. This calculation can, of course, be carried out as explained in the previous article, but it is possible to simplify the equations for E and I so as to greatly reduce the labor of calculation when E and I are to be calculated for many values of l, as follows:

Imagine an "additional length" l' of line to be connected beyond the receiver, the far end of this "additional" line being

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open, and its length l' being so chosen that it is exactly equivalent to the receiver. Then, the given line with the given receiver will be exactly equivalent to the lengthened line with its far end open.

Let $Z_r = E_r/I_r$ be the impedance of the given receiver, and let E_o be the voltage across the distant open end of the lengthened line ($I_o = o$). Then we can use E_o and I_o in equations (iii) and (iv) of Art. 15 in place of E_r and I_r, and E and I so found will then be the correct given values of E_r and I_r. Therefore, I_o being zero, we have

$$E_r = E_o \cosh Ql' \quad \text{(i)}$$

and

$$I_r = \frac{E_o}{U} \cdot \sinh Ql' \quad \text{(ii)}$$

so that

$$E_r/I_r = Z_r = U/\tanh Ql'$$

whence

$$Ql' = \tanh^{-1}(U/Z_r) \quad \text{(iii)}$$

Furthermore, from (i) we have

$$E_o = E_r/\cosh Ql' \quad \text{(iv)}$$

Then the values of E and I at any point on the line distant l from the actual receiver end (or distant $l + l'$ from the fictitious open end) are given by equations (iii) and (iv) of Art. 15 if we put $E_o(= E_r/\cosh Ql')$ for E_r, if we put $I_o = o$ for I_r, and if we put $l + l'$ for l. Therefore

$$E = E_o \cosh Q(l + l') \quad \text{(v)}$$

$$I = \frac{E_o}{U} \cdot \sinh Q(l + l') \quad \text{(vi)}$$

Remark. The added length l' is in general complex, which, being interpreted physically, would mean that the line constants of the added line are not the same as the line constants of the original given line.

17. Wave-length and wave velocity of decaying wave-trains. The wave-length λ of the wave-train which is represented by Fig. 2 is the distance apart of two current (or voltage) vectors for which the phase difference βl is equal to 2π radians. Therefore, putting $\beta\lambda = 2\pi$, we get

$$\lambda = \frac{2\pi}{\beta} \quad \text{(i)}$$

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The velocity V of the wave forms in the wave-train of Fig. 2 is equal to the wave-length divided by the period or to the wave-length multiplied by the frequency f, according to equation (i) of Art. 1 of Chapter II. Therefore

$$V = \frac{2\pi f}{\beta} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

The velocity V is the velocity at which the wave-forms in Fig. 2 would be seen to travel if they were visible. The value of β expressed in terms of R, X, G and B is

$$\beta = +\sqrt{\tfrac{1}{2}[(R^2 + X^2)(G^2 + B^2)]^{\frac{1}{2}} - \tfrac{1}{2}(RG - XB)} \quad \ldots\text{(iii)}$$

where $X = 2\pi fL$ and $B = 2\pi fC$, and, using this value of β in (ii) it may be shown that V is equal to $\sqrt{1/LC}$ under a variety of conditions as follows: 1st when $R = o$ and $G = o$, that is to say, for an ideal line having zero wire resistance and zero leakance, 2nd when the condition of distortionless wave transmission as explained in Art. 23 of Chapter I is satisfied, and 3rd when the frequency is very high. Now an abrupt wave-front, like the front of the ribbon wave in Fig. 16 of Chapter I (a discontinuity of voltage and current values in a wave) means or depends on excessively high-frequency Fourier components in the wave, and therefore the abrupt front of the ribbon wave in Fig. 16 of Chapter I travels at velocity $\sqrt{1/LC}$ even on a line having appreciable wire resistance and leakance.

18. Velocity of wave-forms and velocity of wave-groups. Form velocity and group velocity. The velocity V above defined is the velocity of travel of the individual wave forms, and although V is usually called *wave velocity* it might better be called *form velocity*. **When form velocity varies with wave-length the individual wave-forms in a group of waves travel faster or slower than the group as a whole** as explained in connection with the following figures 9, 10 and 11. Thus we have two distinct velocities associated with waves of a given* wave-length, namely, *form velocity* and *group velocity*.

* The two wave-lengths λ and $\lambda + d\lambda$ in Fig. 9 differ only by an infinitesimal amount.

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This is not a very important matter to the electrical engineer, but it is sufficiently interesting to justify a somewhat detailed discussion.

Group velocity. Let 1, 2, 3, etc., in Fig. 9 be the crests of a train of water waves of wave-length λ traveling at velocity V from the left towards the right, and let a, b, c, etc., be the crests of another train of water waves of wave-length $\lambda + d\lambda$ traveling at velocity $V + dV$ from the left towards the right. The superposition of these two wave-trains gives a succession of wave groups as indicated in Fig. 9. Only one group is shown in Fig. 9. The enveloping curve in Fig. 9 is a curve of sines.

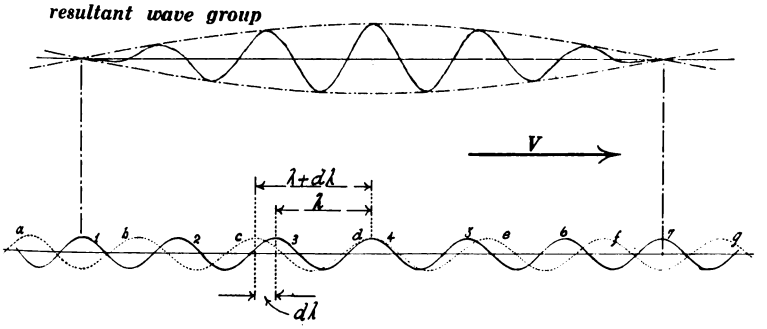

Fig. 9.

The highest crest in the resultant group in Fig. 9 (the crest at the middle of the group) is coincident with the two wave crests 4 and d which are in phase with each other. The velocity of crests a, b, c, etc., *relative* to crests 1, 2, 3, etc., is ΔV. The time required for crest c to overtake crest 3 is $\Delta\lambda/\Delta V$, and when crests c and 3 thus become coincident the center of the group will be at 3 so that during time $\Delta\lambda/\Delta V$ the centre of the group will travel distance λ *backwards* relative to crests 1, 2, 3, etc. Consequently the velocity of the group relative to crests 1, 2, 3, etc., is, $-\lambda/(\Delta\lambda/\Delta V)$ or $-\lambda\cdot\Delta V/\Delta\lambda$, and therefore the velocity G* of the group is

$$G = V - \lambda\frac{dV}{d\lambda} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(iv)}$$

* Not to be confused with the quantity G as listed in Art. 1 of this chapter.

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Only one wave group is shown in Fig. 9 but Fig. 10 is a drawing which shows more than one wave group.

For water waves* of fairly long wave-length (*not* fine ripples whose velocity depends in part on surface tension) the form velocity is

$$V = \sqrt{\frac{g\lambda}{2\pi}} \quad \text{.....(v)}$$

where g is the acceleration of gravity, and therefore, for such water waves equation (iv) gives

$$G = \tfrac{1}{2}V \quad \text{.....(vi)}$$

That is to say, the group velocity of water waves of fairly long

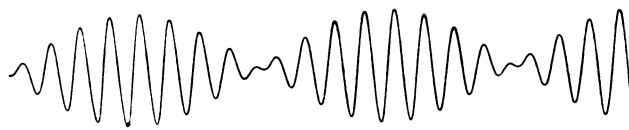

Fig. 10.
Resultant of two wave-trains of the same amplitude but of slightly different wave-lengths.

wave-length is one-half of the form velocity. Each wave in a group like Fig. 9 travels forwards in the group, getting higher and higher as it approaches the middle of the group and dwindling away after it passes the middle of the group. This phenomenon may be seen by watching the bow-waves which are formed by a motor-boat as it runs a straight course on the smooth waters of a lake. Wait until you pass a small floating stick or leaf which will serve to mark a fixed point on the water. Then fix the attention on the portion of the bow-wave system which lies between the dotted lines *a* and *b* in Fig. 11. This portion of the bow-wave system consists of a group of waves

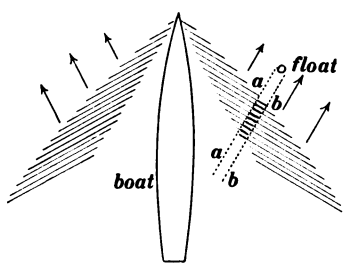

Fig. 11.

* A good discussion of water waves is given in Edser's *General Physics*, pp. 449–481, Macmillan and Co., London, 1920.

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which follow each other at a certain form velocity V, but the group as a whole travels forwards (towards the float) at velocity $\frac{1}{2}V$. The careful observer will see new wave-forms come into existence near the tail end of the group and travel forwards in the group, and each wave-form will be seen to increase in height as it approaches the middle of the group and then dwindle away after it passes the middle of the group. Each new wave-form as it grows in size gets its increasing energy from what is left behind by the wave-forms which preceed it, and each wave form as it dwindles in size as stated gives energy to the waves which follow it. In every case in which the form velocity varies with wave-length the flow of energy in a wave system is very complicated, it is not a simple forward flow or transfer of energy.

The simplest mechanical example which illustrates the distinction between form velocity and group velocity is afforded by the tasseled fringe of a tablecloth which hangs over the edge of a table and in which the individual threads are free. Running the finger along the fringe at *any* velocity V sets the threads vibrating (as pendulums) in such phase relations that wave-forms will be seen to travel along the fringe at velocity V. If one sets a limited portion, only, of the fringe in motion in this way, then wave-forms will travel forwards at velocity V in this portion of the fringe, but the group of wave forms will not travel at all. The group velocity is zero. No energy at all is carried forwards with the traveling wave-forms in this case. There is a close connection between group velocity and the forward transfer of energy by waves, but there is no need to discuss this matter here.

The form velocity of waves on a transmission line (a line which has energy losses) decreases with increasing wave-length and consequently the form velocity is less than the group velocity according to equation (iv).

19. The attenuation constant. The quantity α as defined in Art. 1 of this chapter is called the attenuation constant of a

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line for the given frequency.* The value of α expressed in terms of R, X, G and B is

$$\alpha = +\sqrt{\tfrac{1}{2}[(R^2 + X^2)(G^2 + B^2)]^{\frac{1}{2}} + \tfrac{1}{2}(RG - XB)} \quad \ldots\ldots \text{(i)}$$

The voltage and current in a decaying wave-train are reduced to $1/\gamma$ of their initial values while the wave-forms travel distance l, where $\gamma = \epsilon^{\alpha l}$.

For very low frequency $\alpha = \sqrt{RG}$.

For a distortionless line (see Art. 23 of Chapter I) α has the same value at all frequencies, namely, $\alpha = \sqrt{RG}$.

For very high frequency $\alpha = \sqrt{\tfrac{1}{2}\left(RG + \frac{R^2C}{2L} + \frac{G^2L}{2C}\right)}$, and, of course, R becomes very large at high frequency because of skin effect.

For a given line α increases from $\sqrt{RG}$ at very low frequency to $\sqrt{\tfrac{1}{2}\left(RG + \frac{R^2C}{2L} + \frac{G^2L}{2C}\right)}$, at very high frequency.

20. Wave distortion. Imagine a given non-harmonic electromotive force to act across the home end of a long line. This non-harmonic electromotive force consists of a series of harmonics of increasing frequencies (see discussion of Fourier's series in Chapter VI) and a steady state of the line is established for each harmonic.

At the home end of the line the resultant voltage across the line is, of course, the given non-harmonic voltage, but at greater and greater distances from the home end the wave-form of the resultant voltage across the line will differ more and more from the wave-form of the given non-harmonic voltage across the home end because the high-frequency (short wave-length) Fourier components will be attenuated more than the low-frequency (long wave-length) Fourier components. This change of wave-form is called *wave distortion*, and it is evident from the foregoing statement that the steady state solution of the transmission line as developed in the foregoing articles, when used in conjunction with Fourier's theorem furnishes a complete

* The quantities X and B involve f so that α depends on f.

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solution of wave distortion in the case of periodic waves. This matter is of great importance to the telephone engineer, but a detailed discussion of it is beyond the scope of this text.

21. The use of the steady state solution in the mathematical formulation of transient states. When a harmonic electromotive force begins to act on one end of a transmission line the decaying wave-train which is shown in Fig. 2 does not at once come into existence; theoretically an infinite time must elapse before equations (iii), (iv), (v) and (vii) of Art. 5 are rigorously* true. The steady state is not a transient condition, and, ordinarily, it cannot be used in the mathematical formulation of a transient.

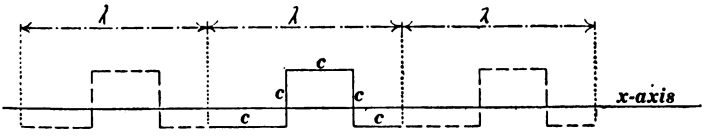

Fig. 12.

The Fourier integral and the use of the steady state in the complete mathematical formulation of any transient. According to Fourier's theorem any periodic curve whatever can be resolved into a series of harmonics. Thus the arbitrary curve *ccccc* in Fig. 12 can be so resolved. But the wave-length λ in Fig. 12 can be as long as you please, and Fourier's theorem can be extended so as to apply to the case in which λ is infinite. This extension of Fourier's theorem is usually designated as *the Fourier integral,* and by means of it any momentary and non-recurring electromotive force whatever can be resolved into an infinite series of harmonic electromotive forces each enduring for all time! This resolution being made, each harmonic component of the electromotive force has its steady-state solution on the line, and the sum of these steady-state solutions is the actual line transient due to the given, non-recurring electromotive force. It is a very remarkable and

* Of course these equations are never rigorously true. See Arts. 2 and 3.

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curious fact that any momentary and non-recurring electromotive force can be resolved into a series of harmonic electromotive forces, as stated; these component electromotive forces add up to zero for all time until the given electromotive force begins to act, they then add up to the given electromotive force while it is acting, and they then add up to zero for all future time! Likewise the corresponding steady states add up to zero everywhere and always until the transient disturbance on the line begins, and their sum dwindles to zero everywhere and always as the transient disturbance dies away. The use of the Fourier integral for the mathematical formulation of line transients is extremely laborious, and any detailed discussion of it is beyond the scope of this text.

PROBLEMS

1. Given a 200-mile, two-wire line with aluminum wires 0·460 inch in diameter spaced 6 feet apart center to center. Calculate the line constants R, X and B for a frequency of 60 cycles per second and find the values of z, ξ, y and ζ. Take $G = o$, use 17·05 as the resistance in ohms of an aluminum wire 0·001 inch in diameter and 1 foot long, and make allowance for skin effect by multiplying the d.c. resistance of the wire by 1·009. Ans. $R = 0{\cdot}8582$ ohms per mile, $X = 1{\cdot}456$ ohms per mile, $B = 2{\cdot}937 \times 10^{-6}$ reciprocal ohms per mile, $z = 1{\cdot}690$ ohms per mile, $\xi = 59{\cdot}28$ degrees, $y = 2{\cdot}937 \times 10^{-6}$ reciprocal ohms per mile, and $\zeta = 90$ degrees.

2. Calculate q, δ, u, and ψ for the line of the previous problem for the frequency of 60 cycles per second. Ans. $q = 0{\cdot}002228$ reciprocal miles, $\delta = 74{\cdot}73$ degrees, $u = 758{\cdot}6$ ohms, $\psi = -15{\cdot}25$ degrees.

3. Calculate the values of α and β for the above line for a frequency of 60 cycles per second. Ans. $\alpha = 0{\cdot}000587$ reciprocal miles, $\beta = 0{\cdot}002149$ reciprocal miles.

4. Given $E_r = 150{,}000$ volts, $i_r = 100$ amperes and $\theta_r = -30$ degrees (I_r being 30 degrees behind E_r in phase), find components of voltage and current at the generator end of the above 200-mile

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transmission line at 60 cycles per second using equations (iii), (iv), (vii) and (viii) of Art. 14. The value of βl is 0·4298 radians or 24·63 degrees.

Answers : Comp. of E_g parallel to E_r = + 165,400 volts
Comp. of E_g 90° ahead of E_r = + 23,980 volts
Comp. of I_g parallel to E_r = + 80·4 amperes
Comp. of I_g 90° ahead of E_r = + 44·17 amperes

Note. The computation of these results, using ordinary tables, took 56 minutes. These results have been checked by duplicate computations and they are dependable, perhaps !

5. Using the results of problems 1, 2 and 3 ; and using the same data as in problem 4 ; calculate the same results as in problem 4, making use of equations (vii), (viii), (ix) and (x) of Art. 15. Use Kennelly's charts I and II as explained in Art. 4 of Appendix D.

Answers : Comp. of E_g parallel to E_r = + 165,300 volts
Comp. of E_g 90° ahead of E_r = + 24,200 volts.
Comp. of I_g parallel to E_r = + 79·9 amperes
Comp. of I_g 90° ahead of E_r = + 44·3 amperes

Note. The computation of the answers to this problem, using four-place logarithms, took 30 minutes, and this time could have been reduced very considerably by using a slide rule. The slide rule is certainly permissible because slide-rule inaccuracies are very considerably less than the inaccuracies involved in the use of the Kennelly charts. Another important consideration is that mistakes in computation are much less likely to occur in the computations of this problem than in the computations of problem 4, and the results are more easily checked.

The answers to problem 4 were calculated using four-place logarithms and the answers to problem 4 are therefore dependable, if carefully checked, to about 1/40 of one per cent. Of course this degree of precision is physically meaningless because of the uncertainty of the line constants, but the precise calculations of problem 4 enable us to form some estimate of the precision of the answers to problem 5.

The discrepancies between the answers to problems 4 and 5 are due almost wholly to the inaccuracies in the use of the Kennelly charts. Thus the component of I_g parallel to E_r as calculated in problem 5 differs by 0·5 ampere from its value as calculated in problem 4, and this difference is about ½ of 1 per cent of the magnitude of I_g.

By taking great care in the use of the Kennelly charts results can be obtained which are dependable to about ½ of 1 per cent, and this degree of precision is sufficient for most practical purposes.

6. Using the value of β as given by equation (iii) of Art. 17 and using $\lambda = 2\pi/\beta$, derive an expression for f in terms of λ. Use this expression for f in $V = f\lambda$ thus getting an expression for form

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velocity in terms of λ. Then derive an expression for the group velocity for wave-length λ. Take leakage conductance G of the line as zero for the sake of simplicity. Ans. group velocity

$$= (8\pi R^2C^2\lambda^2 + 64\pi^3LC)/(R^2C^2\lambda^2 + 16\pi^2LC)^{3/2}.$$

7. Prove that the attenuation constant α for a distortionless line is independent of the frequency and equal to $\sqrt{RG}$.

8. Prove that the attenuation constant α for infinite frequency is as stated in Art. 19.

9. A 500-mile telephone line, without inductive loading, consists of two copper wires each $\frac{1}{8}$ inch in diameter spaced 2 feet apart center to center. Neglecting skin effect and line leakage, find the attenuation constant α for a frequency of 300 cycles per second and for a frequency of 1500 cycles per second; and find the attenuation ratio of the entire line for each frequency. Ans. $\alpha = 0{\cdot}00449$ for 300 cycles per second; $\alpha = 0{\cdot}00485$ for 1500 cycles per second; voltage and current at the distant end of the line are reduced to 0·106 of their home-end values for a frequency of 300 cycles per second and to 0·088 of their home-end values for a frequency of 1500 cycles per second.

10. Calculate the form velocity for decaying wave-trains on the line of the previous problem for frequencies of 3, 300 and 1500 cycles per second and for infinite frequency. Ans. 26,600 miles per second; 170,200 miles per second; 185,400 miles per second; 186,000 miles per second.

11. The abrupt front *bd* of the ribbon wave in Fig. 16 of Chapter I is 1000 volts as it starts out from the home-end of the line. Find the voltage in the abrupt front 0·001 second after closing the switch and state how far the abrupt front has traveled during the 0·001 second. Use line data from problem 9, and neglect great increase of wire resistance due to skin effect.

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CHAPTER IV

THE THREE-WIRE, THREE-PHASE LINE AND THE CIRCLE DIAGRAM

1. Specifications of three independent two-wire single-phase lines which would be equivalent to a given three-wire three-phase line. It is desired to apply the theory of the steady state of the two-wire single-phase line, as developed in Chapter III, to the three-wire three-phase line; and to do this it is necessary to define (to specify the constants for) three independent two-wire single-phase lines which would be together equivalent to the given three-wire three-phase line. This procedure depends on the introduction of a fictitious neutral wire N as indicated in Fig. 1.

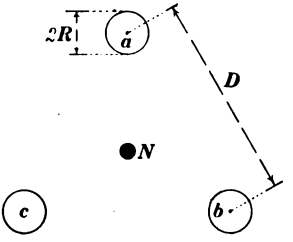

Fig. 1.

a, b and c are the given line wires and N is the fictitious neutral wire. Distance between centers of wires D, radius of each wire R.

The three single-phase lines can be equivalent to the given three-wire three-phase line only when the three-phase system is balanced,* and under these conditions the central neutral wire N could be actually installed without altering the voltages or the currents anywhere. There would be no current in the neutral wire and therefore no resistance drop of voltage along it. No magnetic flux would encircle the neutral wire and there would be no reactance drop of voltage along it. No electric charges would exist on the neutral wire and there would be no need of charging-current in it. If the three given line wires

* And when electromotive forces and currents are without harmonics of orders divisible by 3.

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a, *b* and *c* are similarly insulated no leakage current would flow to or from the neutral wire. *Therefore for each of our equivalent single-phase lines we will take one of the given line wires and a fictitious neutral NN of zero resistance* as indicated in Fig. 2, and we will determine the constants R, L, G and C which this fictitious two-wire line must have. See Art. 1 of Chapter III for explanation of notation.

(i) Current in wire *a* in Fig. 2 is to be exactly the same as the current in wire *a* in Fig. 1, that is to say, the load current is to be the same, the charging current is to be the same, and the leakage current from *a* to *N* in Fig. 2 is to be the same as the leakage current which leaves wire *a* in Fig. 1.

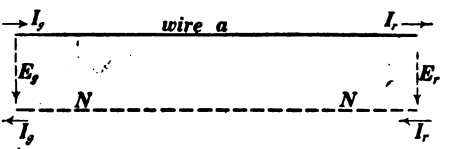

Fig. 2.

Single-phase element of a three-wire three-phase line. E_g is the Y-voltage of the three-phase line at the generator end and E_r is the Y-voltage of the three-phase line at the receiver end, and, as this figure shows, E_g and E_r are, respectively, the generator and receiver voltages of the single-phase line.

(ii) The resistance drop of voltage over the pair of wires in Fig. 2 is to be the same as the resistance drop of voltage along the one wire *a* in Fig. 1. This condition is most easily met by thinking of wire *a* as having the same resistance in Figs. 1 and 2 and assuming that wire *N* has zero resistance.

(iii) The reactance drop of voltage along the pair of wires in Fig. 2 must be the same as the reactance drop of voltage along the single wire *a* in Fig. 1. This condition requires the inductance per unit length, L, of the fictitious two-wire line in Fig. 2 to be equal to $\frac{1}{2}L'$, where L' is the inductance per unit length of *any pair* of the line wires in Fig. 1. See Art. 7 of Appendix A.

(iv) The charging current (out on *a* and back on *N*, let us say) in Fig. 2 must be the same as the charging current in wire *a* in Fig. 1. This condition requires the capacity per unit

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length C of the fictitious *pair of wires* in Fig. 2 to be equal to $2C'$, where C' is the capacity per unit length of *any pair* of the line wires in Fig. 1. See Art. 9 of Appendix A.

(v) The leakage current from wire a to wire N in Fig. 2 must be equal to the leakage current from wire a in Fig. 1. This requires the leakage conductance G between the wires of unit length of the fictitious line in Fig. 2 to be equal to $2G'$ where G' is the measured leakage conductance between any pair of the line wires in Fig. 1.

As a matter of fact the leakage conductance between wires of a well constructed air line is always negligible, and it is not worth while therefore to show why $G = 2G'$.

(vi) The single-phase line in Fig. 2 is to transmit one-third of the power transmitted by the three-phase line in Fig. 1, the current in wire a in Fig. 2 is the Y-current of the three-phase line and the voltage across the line of Fig. 2 at any point is the Y-voltage of the three-phase line at that point.

2. Algebraic versus graphical line calculations. The calculation of generator voltage, current and power factor for given values of receiver voltage, current and power factor by the straightforward use of the equations developed in Chapter III is tedious, and to calculate such results for a whole series of values of receiver voltage, current and power factor would be extremely tedious. Fortunately, however, these calculations can be made with sufficient precision for most practical purposes by a graphical method based on a clock-diagram in which E_r, I_r, E_g, I_g, etc., are shown as vectors. This clock-diagram is called *the circle diagram* of the transmission line* because of the great importance in the diagram of five or six systems of circular loci, and the circle diagram is important not only because of its use in obtaining numerical results but also because it enables one to see at a glance many things

* Evans and Sels, *Electric Journal*, 1921; articles beginning on pages 306, 356, 419 and 530.

L. Thieleman, "Graphical Calculation of Long Transmission Lines," *Revue Générale de l'Électricité*, September, 1920, to May, 1921.

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which would be very difficult to infer from an extensive tabulation of calculated numerical results.

It is the purpose of this chapter to explain in full some of the simpler properties of the circle diagram and to set forth the practical rules for constructing and using the diagram.

The circle diagram has to do with the single-phase transmission line, and methods for adapting the diagram to the three-wire three-phase line when the three-phase system is balanced are given in Art. 1 of this chapter.

3. The generalised transmission line.—The circle diagram applies to a generator and a receiver connected by *any network whatever*, and it is important to develop the diagram for this general case because in practice the simple transmission line is always combined with step-up and step-down transformers and other regulating devices.

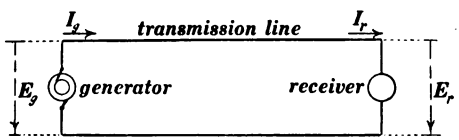

Fig. 3.

Let the two lines in Fig. 3 represent any network whatever connecting a generator and a receiver. It can be shown that for the steady state*

$$\left.\begin{aligned} E_p &= AE_r \\ I_p &= CE_r \end{aligned}\right\} \quad \text{.....(i)}$$

and

where E_p and I_p are generator voltage and current with *open* receiver ($I_r = 0$), where E_r is any given receiver voltage, and where A and C are complex constants. It is helpful to associate the subscript p with the letter p in the word *open*.

Also it can be shown that

$$\left.\begin{aligned} E_s &= BI_r \\ I_s &= DI_r \end{aligned}\right\} \quad \text{.....(ii)}$$

and

where E_s and I_s are generator voltage and current with

* Voltages and currents everywhere of precisely the same frequency phase relations everywhere fixed and root-of-mean-square values everywhere constant.

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short-circuited receiver ($E_r = 0$), where I_r is any given receiver current, and where B and D are complex constants. It is helpful to associate the subscript s with the letter s in the word *short-circuited*.

It is not worth while to derive equations (i) and equations (ii), but it is interesting to note that equations (i), for example, are exactly equivalent to the statements that E_p and E_r have a fixed ratio and a fixed phase difference, and that I_p and E_r have a fixed ratio and a fixed phase difference; and these two statements are essentially equivalent to the definition of the steady state.

The condition or state set forth by equations (i) and the condition or state set forth by equations (ii) may be superposed exactly as in Arts. 12 and 13 of Chapter III, and, using E_g for $E_p + E_s$ and I_g for $I_p + I_s$, we get

$$E_g = AE_r + BI_r \quad \text{(iii)}$$
$$I_g = CE_r + DI_r \quad \text{(iv)}$$

These two equations, A, B, C and D being complex constants, express the relations between generator voltage and current and receiver voltage and current when generator and receiver are connected by any network whatever.*

The four complex constants A, B, C and D of the transmission network must be known if one is to construct the circle diagram for the network, and it is most convenient to express these constants in polar forms as follows:

$$A = a\underline{/\alpha} \quad \text{(v)}$$
$$B = b\underline{/\beta} \quad \text{(vi)}$$
$$C = c\underline{/\gamma} \quad \text{(vii)}$$
$$D = d\underline{/\delta} \quad \text{(viii)}$$

Remark. The notation of this chapter is the same as the notation of Chapter III except for the constants A, B, C and D and the related quantities a, b, c, d, α, β, γ and δ. Unfortunately some of these symbols have already been used in Chapter III, and in retaining the notation of Chapter III we

* Of course electromotive forces and currents are assumed to be harmonic, and actions such as are described in Art. 4 of Chapter VII are ignored.

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are led in this chapter to use B, C, c, α and β sometimes in accordance with Chapter III and sometimes in accordance with the above equations (iii) to (viii).

Example 1. Figure 4 shows a generator and receiver connected through a simple impedance Z $(=r + jx)$. Here we have

$$E_g = E_r + ZI_r$$

and

$$I_g = I_r$$

whence, comparing these equations with equations (iii) and (iv), we get $A = 1$, $B = Z$, $C = 0$ and $D = 1$.

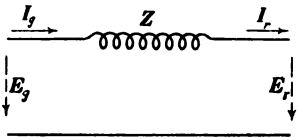

Fig. 4.

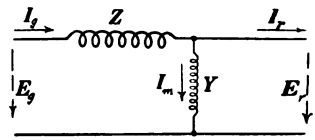

Fig. 5.

Example 2. Figure 5 shows the same as Fig. 4 but with an admittance Y connected across the receiver. In this case we have

$$E_g = E_r + ZI_g$$

and

$$I_g = I_m + I_r = YE_r + I_r$$

and from these equations it is easy to get

$$E_g = (1 + ZY)E_r + ZI_r$$

and

$$I_g = YE_r + I_r$$

so that $A = 1 + ZY$, $B = Z$, $C = Y$ and $D = 1$.

Example 3. A simple transmission line without transformers or other regulating devices. In this case E and I as given by equations (iii) and (iv) of Art. 15, Chapter III, are E_g and I_g if for l we use the full length of the line. Therefore we get

$$A = \cosh Ql \quad \text{(ix)}$$
$$B = U \sinh Ql \quad \text{(x)}$$
$$C = (\sinh Ql)/U \quad \text{(xi)}$$
$$D = A \quad \text{(xii)}$$

and having found the polar expressions for U, $\cosh Ql$ and $\sinh Ql$, namely, $U = u\underline{/\psi}$, $\cosh Ql = c^*\underline{/c'}$ and $\sinh Ql = s\underline{/s'}$ as in Art. 15 of Chapter III, we have: $a = c^*$, $\alpha = c'$, $b = us$,

* The symbol c^* stands for the numerical part of $\cosh Ql$ and c stands for the numerical part of the complex constant C in equation (vii). See remark following equation (viii). Note that s and s' here denote the parts of the polar expression for $\sinh Ql$, whereas s and s' are used later in this chapter for voltage and current scales in the circle diagram.

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$\beta = \psi + s'$, $c = s/u$, $\gamma = s' - \psi$, $d = a$ and $\delta = \alpha$. It is to be noted that $A = D$ for any transmission network which is alike at both ends.

To calculate the values of a, b, c, d, α^*, β^*, γ *and* δ by using ordinary trigonometric and logarithmic tables, we may use equations (i) and (v) of Art. 14, Chapter III, from which we get

$$A = \frac{1}{2}\left[\epsilon^{\alpha l}\cos\beta l + \epsilon^{-\alpha l}\cos\beta l\right] + \frac{j}{2}\left[\epsilon^{\alpha l}\sin\beta l - \epsilon^{-\alpha l}\sin\beta l\right] \qquad \text{(xiii)}$$

$$B = \frac{u}{2}\left[\epsilon^{\alpha l}\cos(\beta l + \psi) - \epsilon^{-\alpha l}\cos(\beta l - \psi)\right] + \frac{ju}{2}\left[\epsilon^{\alpha l}\sin(\beta l + \psi) + \epsilon^{-\alpha l}\sin(\beta l - \psi)\right] \ldots \text{(xiv)}$$

$$C = \frac{1}{2u}\left[\epsilon^{\alpha l}\cos(\beta l - \psi) - \epsilon^{-\alpha l}\cos(\beta l + \psi)\right] + \frac{j}{2u}\left[\epsilon^{\alpha l}\sin(\beta l - \psi) + \epsilon^{-\alpha l}\sin(\beta l + \psi)\right] \ldots \text{(xv)}$$

$$D = A \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(xvi)}$$

Let A, for example, be for the moment represented by $x + jy$. Then the values of x and y are given in equation (xiii), and we get $a = \sqrt{x^2 + y^2}$ and $\alpha^* = \tan^{-1}(y/x)$.

Example 4. *A simple transmission line with unlike transformers at its ends for step-up and step-down transformation.* The central portion of Fig. 6 represents the simple transmission line and the end portions of Fig. 6 show the simple-circuit equivalents of the transformers. We will let A, B, C and D be the complex constants for the transmission line itself in Fig. 6, and we will let A_o, B_o, C_o and D_o be the complex constants for the entire arrangement in Fig. 6.

To find A_o it is only necessary to set up the equation $E_p = A_o E_r$ where E_p is the generator voltage for given receiver voltage E_r when the receiver current I_r is zero. This is evident from the first of equations (i) above. Therefore, beginning at

* Note that α^* and β^* designate the angular parts of A and B. See, equations (v) and (vi), above. The symbols α and β in (xiii), (xiv) and (xv) are used in accordance with Chapter III.

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the receiver end in Fig. 6 with $I_r = 0$ we get $M_r = Y_r E_r$ where M_r is the current through Y_r, and then, quite simply, we get

$$E_r' = (1 + T_r Y_r)E_r$$

and

$$I_r' = Y_r E_r$$

But from equations (iii) and (iv) we have

$$E_g' = AE_r' + BI_r'$$

and

$$I_g' = CE_r' + DI_r'$$

so that, using the above values for E_r' and I_r', we get

$$E_g' = A(1 + T_r Y_r)E_r + BY_r E_r$$

and

$$I_g' = C(1 + T_r Y_r)E_r + DY_r E_r$$

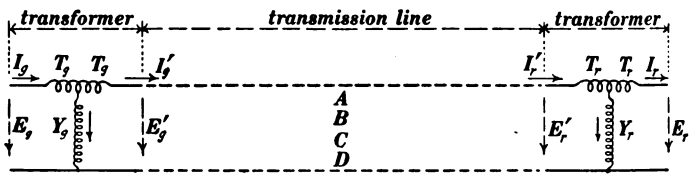

Fig. 6.

T is here *one-half* of the short-circuit impedance of the transformer supplied from the high-voltage side, and Y is the admittance of the transformer supplied from high-voltage side with low-voltage side on open-circuit.

E_r can, of course, have any value, but when T and Y are taken as here defined the proper value of E_r is the high-voltage which is suitable for the high-voltage side of the receiver transformer.

Under the assumed conditions, namely that $I_r = 0$, we have $E_p + A_o E_r$ and from Fig. 6 it is easy to write out an expression for E_p, namely

$$A_o E_r = E_g' + T_g I_g' + T_g[I_g' + Y_g(E_g' + T_g I_g')]$$

and by using the above values of E_g' and I_g' and cancelling E_r we get

$$A_o = A[(1 + T_r Y_r)(1 + T_g Y_g) + 2T_g Y_r(1 + \tfrac{1}{2}T_g Y_g)] + BY_r(1 + T_g Y_g) + 2CT_g(1 + T_r Y_r)(1 + \tfrac{1}{2}T_g Y_g)$$

To get the values of a_o and α_o this expression must be reduced to the form $A_o = x + jy$ then $a_o = \sqrt{x^2 + y^2}$ and $\alpha_o = \tan^{-1} y/x$.

The accompanying table gives the expressions for A_o, B_o, C_o and D_o for a variety of cases. The quantity T in this table

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is *one-half* of the short-circuit impedance of a transformer when supplied from the high-voltage side, and the subscripts r and g are used to distinguish between transformers at the generator and receiver ends of the line.

TABLE*

COMPLEX QUANTITY CONSTANTS OF TRANSMISSION NETWORKS

Note. A, B, C and D represent the complex constants for a simple line. See equations (ix), (x), (xi) and (xii).

(i) Simple line with impedance Z_r in series at receiver end.

$$A_0 = A \qquad C_0 = C$$
$$B_0 = B + AZ_r \qquad D_0 = A + CZ_r$$

(ii) Simple line with impedance Z_g in series at generator end.

$$A_0 = A + CZ_g \qquad C_0 = C$$
$$B_0 = B + AZ_g \qquad D_0 = A$$

(iii) Simple line with impedance Z_r in series at receiver end and impedance Z_g in series at generator end.

$$A_0 = A + CZ_g$$
$$B_0 = B + A(Z_r + Z_g) + CZ_rZ_g$$
$$C_0 = C$$
$$D_0 = A + CZ_r$$

(iv) Simple line with transformer at receiver end, Y_r being admittance of transformer on open circuit when supplied from high-voltage side, and $2T_r$ being the short-circuit impedance of the transformer when supplied from the high-voltage side.

(*a*) *Exact formulas.*

$$A_0 = A(1 + T_rY_r) + BY_r$$
$$B_0 = B(1 + T_rY_r) + AT_r(2 + T_rY_r)$$
$$C_0 = C(1 + T_rY_r) + AY_r$$
$$D_0 = A(1 + T_rY_r) + CT_r(2 + T_rY_r)$$

(*b*) *Approximate formulas, precision about* $\pm$ 0·5%

$$A_0 = A + BY_r$$
$$B_0 = B + 2AT_r$$
$$C_0 = C + AY_r$$
$$D_0 = A + 2CT_r$$

(v) Simple line with transformer at generating end.

(*a*) *Exact formulas.*

$$A_0 = A(1 + T_gY_g) + CT_g(2 + T_gY_g)$$
$$B_0 = B(1 + T_gY_g) + AT_g(2 + T_gY_g)$$
$$C_0 = C(1 + T_gY_g) + AY_g$$
$$D_0 = A(1 + T_gY_g) + BY_g$$

* Taken from the series of articles by Evans and Sels, *Electric Journal*, 1921, pages 306, 356, 419 and 530.

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(*b*) *Approximate formulas, precision about* $\pm 0{\cdot}5\%$.

$$A_o = A + 2CT_g$$
$$B_o = B + 2AT_g$$
$$C_o = C + AY_g$$
$$D = A + BY_g$$

(vi) Simple line with transformers at both ends as shown in Fig. 6.

(*a*) *Exact formulas.*

$$A_o = A[(1 + T_rY_r)(1 + T_gY_g) + T_gY_r(2 + T_gY_g)] + BY_r(1 + T_gY_g) + CT_g(1 + T_rY_r)(2 + T_gY_g)$$
$$B_o = B(1 + T_rY_r)(1 + T_gY_g) + A[T_r(2 + T_rY_r)(1 + T_gY_g) + T_g(1 + T_rY_r)(2 + T_gY_g)] + CT_rT_g(2 + T_rY_r)(2 + T_gY_g)$$
$$C_o = C(1 + T_rY_r)(1 + T_gY_g) + A[Y_r(1 + T_gY_g) + Y_g(1 + T_rY_r)] + BY_rY_g$$
$$D_o = A[(1 + T_rY_r)(1 + T_gY_g) + T_rY_g(2 + T_rY_r)] + BY_g(1 + T_rY_r) + CT_r(2 + T_rY_r)(1 + T_gY_g)$$

(*b*) *Approximate formulas, precision about* $\pm 1\%$.

$$A_o = A + 2CT_g + Y_r(2AT_g + B)$$
$$B_o = B + 2A(T_r + T_g) + 4CT_rT_g$$
$$C_o = C + BY_rY_g + A(Y_r + Y_g)$$
$$D_o = A + 2CT_r + Y_g(2AT_r + B)$$

4. Clock-diagrams representing equations (iii) and (iv) of Art. 3. Let the line OR in Fig. 7 represent E_r and let the line OI_r represent a lagging current I_r. Then OF represents AE_r, FM represents BI_r and OM represents E_g. To multiply the vector E_r by A turn the vector E_r forwards through the angle α and multiply the numerical value of E_r by a according to equation (v) of Art. 3; and to multiply I_r by B turn I_r forwards through the angle β and multiply the numerical value of I_r by b according to equation (vi) of Art. 3.

Let the line $O'R'$ in Fig. 8 represent E_r and let the line $O'I_r$ represent a lagging current I_r. Then $O'F'$ represents CE_r, $F'M'$ represents DI_r and $O'M'$ represents I_g. To multiply E_r by C turn E_r forwards through the angle γ and multiply the numerical value of E_r by c according to equation (vii) of Art. 3; and to multiply I_r by D turn I_r forwards through the angle δ and multiply the numerical value of I_r by d according to equation (viii) of Art. 3.

It is to be particularly noted that I_r has been turned β radians forwards to give FM in Fig. 7 and that I_r has been turned δ radians forwards to give $F'M'$ in Fig. 8. **Therefore**

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$F'M'$ in Fig. 8 would have to be turned $\beta - \delta$ radians forwards to bring it into the direction of FM in Fig. 7.

Let s volts per inch be the arbitrarily chosen scale to which the line FM represents the voltage BI_r and let it be required to find the scale s' in amperes per inch to which the line $F'M'$ must represent the current DI_r so that the lines FM and $F'M'$ may be of the same length L. Now the numerical value of BI_r is bi_r volts and the numerical value of DI_r is di_r amperes, so that, evidently, we must have $bi_r = sL$ and $di_r = s'L$,

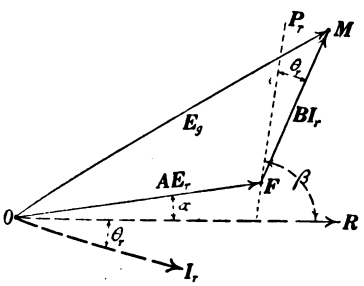

Fig. 7.

Heavy lines are current vectors.
Light lines are voltage vectors.

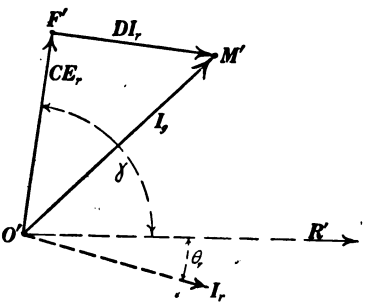

Fig. 8.

Heavy lines are current vectors.
Light lines are voltage vectors.

whence, dividing one of these equations by the other member by member, we get

$$s' = \frac{d}{b}.s \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

5. Combining of Figs. 7 and 8 to give the circle diagram. Suppose Figs. 7 and 8 to be drawn to such scales as to make the lines FM and $F'M'$ of the same length as above explained, suppose that Fig. 8 be turned $\beta - \delta$ radians forwards (counter clockwise or in the conventional direction of clock-diagram rotation) so as to bring $F'M'$ into the direction of FM as above explained, and suppose the two figures to be joined together so as to bring the two points F and F' into coincidence and the two points M and M' into coincidence as shown in Fig. 9; then **Fig. 9 is the desired circle diagram.**

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The circle diagram is especially useful for the case in which the receiver voltage is constant. Therefore throughout the following discussion E_r is supposed to have a known constant value. Later a circle diagram will be constructed for E_g constant, and eventually these two circle diagrams will be modified so that they can be used for any given values of E_r and E_g.

Representation of receiver current I_r in the circle diagram. The vector E_g in Fig. 9 shows the phase and value to E_g (s volts per inch), and the vector I_g shows the value of I_g to the scale

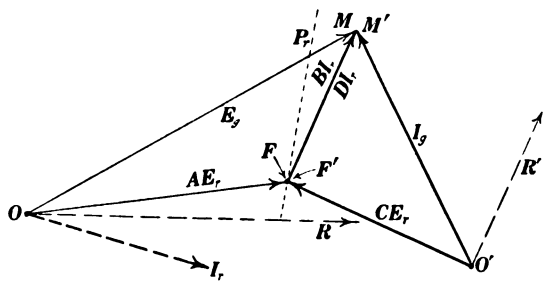

Fig. 9.

The fine lines are voltage vectors and they show the true phases of the voltages relative to OR which represents E_r.

The heavy lines are current vectors and they have all been turned $\beta - \delta$ radians forwards.

Hereafter the point FF' is designated by F and the point MM' is designated by M.

of s' ($= ds/b$) amperes per inch; furthermore the vector I_g shows the true phase of I_g if we make allowance for the fact that all of the current vectors in Fig. 9 have been turned $\beta - \delta$ radians forwards from their positions in Fig. 8.

It is highly desirable to find a vector in Fig. 9 which will show the value and phase of I_r. Now the vector $F'M'$ shows the value of the current DI_r to the scale of s' amperes per inch, or, since the numerical value of DI_r is dI_r amperes, it is evident that $F'M'$ shows the value of I_r to the scale of $s'/d(= s/b)$ amperes per inch. Furthermore $FM(= F'M')$ is β radians ahead of I_r because it is equal to BI_r; therefore FP_r must be

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β radians ahead of E_r if the angle between FP_r and FM is to be θ_r. Note that the angle θ_r in Fig. 10 is an angle of lag of I_r behind E_r and it is therefore considered as negative. See Art. 1 of Chapter III.

Why E_r must be constant if the circle diagram is to be useful. This is because the points F and O' are fixed with reference to the point O and the line OR (for a given transmission network) when E_r is constant. The only variables in the diagram, when E_r is constant, are the position of the point

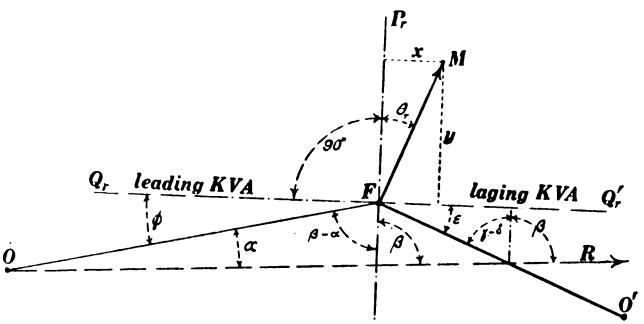

Fig. 10.

ϕ stands for $90° - \beta + \alpha$.

$O'F'$ has been turned γ radians forwards from $O'R'$ in Fig. 8, and it has been turned $\beta - \delta$ farther forwards here and in Fig. 9. Therefore $O'F$ is here $\beta + \gamma - \delta$ ahead of OR. Therefore ϵ stands for $90° - \gamma + \delta$.

M and the quantities E_g, I_g, I_r, etc., which depend on the location of M. When the location of M is given the whole state of things is completely determined. The point M is therefore called the *operating point* of the diagram.

Representation of the power P_r delivered to the receiver and of the wattless volt-amperes Q_r at the receiver. The two components x and y of the line FM in Fig. 10 represent $I_r \sin \theta_r$ and $I_r \cos \theta_r$, respectively, to the scale of s'/d or s/b amperes per inch, as explained above, so that $I_r \sin \theta_r = \frac{s}{b} \cdot x$ and $I_r \cos \theta_r = \frac{s}{b} \cdot y$, whence, multiplying both members of each

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of these equations by E_r, we get

$$P_r = E_r I_r \cos \theta_r = \frac{E_r s}{b} \cdot y$$

and

$$Q_r = E_r I_r \sin \theta_r = \frac{E_r s}{b} \cdot x$$

That is to say, the two co-ordinates x and y of the point M represent Q_r and P_r respectively to the scale $E_r s/b$ watts or volt-amperes per inch.

Determination of the angle θ_g (angle by which I_g leads E_g in phase) **from the circle diagram.** The angle OMO' in Fig. 9 is evidently the angle from the vector E_g forwards to the vector I_g; but the vector I_g has been turned $\beta - \delta$ radians forwards in the construction of the diagram. Therefore the angle OMO' in the diagram is equal to $\theta_g + (\beta - \delta)$, so that, if the angle OMO' is measured off in the diagram, the value of θ_g can be found since β and δ are known.

Similarly in the "generator end" diagram, Fig. 12, the angle OM_gO_g' (the point M_g is the operating point in Fig. 12 and it may be anywhere in the diagram) is evidently the angle from OM_g ($= E_r$) backwards to $O_g'M_g (= I_r)$, but I_r has been turned $\beta + \alpha$ radians backwards in the construction of the diagram so that $\theta_r = \beta + \alpha - OM_gO_g'$, θ_r being considered as positive when I_r is ahead of E_r in phase.

The easily evident properties of the circle diagram are as follows:

(*a*) The loci of M for constant values of I_r are circles centering at F.

(*b*) The loci of M for constant values of E_g are circles centering at O.

(*c*) The loci of M for constant values of I_g are circles centering at O'.

(*d*) The loci of M for constant values of P_r are straight lines parallel to Q_rF.

(*e*) The loci of M for constant values of Q_r are straight lines parallel to P_rF.

(*f*) The loci of M for constant values of receiver power

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factor (for constant values of θ_r) are straight lines radiating from the point F*.

6. Circle diagram for constant generator voltage E_g. For the sake of simplicity we will call the circle diagram for constant receiver voltage E_r the "receiver-end" diagram, and the circle diagram for constant generator voltage E_g we will call the "generator-end" diagram.

When the co-ordinate y of the operating point M in Fig. 10 is negative, that is to say, when M is below the line Q_rQ_r', the value of P_r is negative and the "receiver" is in fact a generator; then, E_r being constant, it is evident that Fig. 10 is a circle diagram for constant generator voltage. When Fig. 10 is so used one is greatly confused by the fact that E_r, I_r, θ_r, P_r and Q_r refer to what is in fact a generator. It is better, to avoid this confusion, to transform equations (iii) and (iv) of Art. 3 so as to express E_r and I_r in terms of E_g and I_g and proceed exactly as in Arts. 4 and 5 to set up a circle diagram for constant E_g (a "generator-end" diagram). By transforming† equations (iii) and (iv) of Art. 3 we get

$$E_r = DE_g - BI_g \quad \text{(i)}$$

and

$$I_r = -CE_g + AI_g \quad \text{(ii)}$$

and on the basis of these equations a circle diagram for constant E_g can be constructed as explained in Art. 8.

7. Directions for constructing the "receiver-end" diagram. Using good cross-section paper, lay down values of P_r along the axis F_rP_r (see Fig. 11), and lay down values of *leading* and *lagging* receiver volt-amperes along F_rQ_r and along F_rQ_r', respectively, using same scale for P_r and Q_r. Then calculate

* Other interesting loci are; (*g*) the loci of M for constant values of generator power factor (constant values of θ_g), (*h*) the loci of M for constant values of the generator power output P_g, and (i) the loci of M for constant values of the efficiency P_r/P_g. These loci (*g*), (*h*) and (*i*) are all circles and they are discussed in detail in Thieleman's original papers (see footnote to Art. 2), but they are not of sufficient practical importance to be discussed here.

† This transformation is complicated by the fact that A, B, C and D are not independent. It can be shown, however, that the relation $AD - BC = 1$ is always satisfied, and, this being the case, equations (iii) and (iv) of Art. 3 give the above equations (i) and (ii).

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the P_r and Q_r co-ordinates of the points O and O_r' and locate these points as shown in Fig. 11. The circle diagram is then essentially complete.

$$\left.\begin{aligned} P_r\text{-co-ordinate of } O &= -\frac{a}{b}\cdot e_r^2 \cos(\beta - \alpha) \text{ "watts"} \\ Q_r\text{-co-ordinate of } O &= +\frac{a}{b}\cdot e_r^2 \sin(\beta - \alpha) \text{ "watts"} \\ P_r\text{-co-ordinate of } O_r' &= -\frac{c}{d}\cdot e_r^2 \cos(\gamma - \delta) \text{ "watts"} \\ Q_r\text{-co-ordinate of } O_r' &= -\frac{c}{d}\cdot e_r^2 \sin(\gamma - \delta) \text{ "watts"} \end{aligned}\right\} \quad \text{(i)}$$

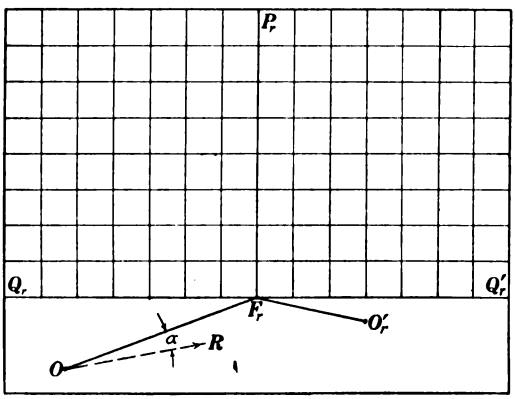

Fig. 11.

"Receiver end" diagram. A positive value of P_r (upwards) means power delivered to the receiver. Leading volt-amperes is measured towards the left.

It is sufficient to show the derivation of one only of these co-ordinates. Let us therefore derive the expression for the P_r-co-ordinate of O. Looking at Fig. 10 it is evident that this co-ordinate is equal to $-\overline{FO}\sin(90° - \beta + \alpha)$, or, using ae_r for the value of OF in volts, we get $-ae_r \sin(90° - \beta + \alpha)$ or $-ae_r \cos(\beta - \alpha)$ for the value of the co-ordinate in volts; but the scale of volts in Fig. 10 is b/e_r volts per "watt" so that the value of the co-ordinate in volts must be multiplied by e_r/b to give its value in "watts."

The various scales in the circle diagram of Fig. 10 are

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brought together in the accompanying table for convenience. Note that if the voltage scale is s volts per inch, the scale for I_g is sd/b amperes per inch, where b is expressible in ohms so that sd/b is expressible in reciprocal ohms. All lines in a consistent clock-diagram must represent quantities of the same kind, and this condition is realized in the circle diagram by virtue of the physical dimensions of the various scale numbers.

TABLE

SCALES IN THE CIRCLE DIAGRAM

In the "receiver-end" diagram; E_r constant:

1. When the voltage scale, s volts per inch, is arbitrarily chosen.

$O'M_r$ represents I_g ds/b amperes per inch.
FM_r represents I_r s/b amperes per inch.
Scale for P_r and Q_r $e_r s/b$ watts or volt-amp. per inch.

2. When scale for P_r and Q_r is arbitrarily chosen and when lengths of vectors are expressed in co-ordinate units or "watts."

OM_r represents E_g b/e_r volts per "watt."
$O'M_r$ represents I_g d/e_r amperes per "watt."
FM_r represents I_r $1/e_r$ amperes per "watt."

In the "generator-end" diagram; E_g constant.

1. When the voltage scale, s volts per inch, is arbitrarily chosen.

$O'M_g$ represents I_r as/b amperes per inch.
FM_g represents I_g s/b amperes per inch.
Scale for P_g and Q_g $e_g s/b$ watts or volt-amp. per inch.

2. When scale for P_g and Q_g is arbitrarily chosen and when lengths of vectors are expressed in co-ordinate units or "watts."

OM_g represents E_r b/e_g volts per "watt."
$O'M_g$ represents I_r a/e_g amperes per "watt."
FM_g represents I_g $1/e_g$ amperes per "watt."

Remark 1. M_r is the operating point in the "receiver-end" diagram and M_g is the operating point in the "generator-end" diagram.

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Remark 2. The "watt" scales are easily derivable from the "inch" scales. For example, let L be the length of OM_r in inches then

$$E_g = sL \quad \text{(ii)}$$

Let W be the "length" of OM_r in "watts," that is to say, W is the number of watts represented by L inches, so that

$$W = \frac{e_r s}{b} \cdot L \quad \text{(iii)}$$

Then, putting the value of L from (iii) in (ii), we get

$$E_g = \frac{b}{e_r} \cdot W \quad \text{(iv)}$$

which, being interpreted, means that the "length" W of OM_r in "watts" must be multiplied by b/e_r to give the value of E_g in volts.

Remark 3. It is sometimes convenient and helpful to draw the circles of constant voltage and the circles of constant current which center at O and at O' respectively, and the radius of any one of these circles for a given value of voltage or current is easily found from the scales which are given in the above table. Thus for a given value of E_g in the "receiver end" diagram the radius in "watts" is $E_g e_r/b$.

Remark 4. The "reduced" receiver-end diagram is constructed for $e_r = 1$ volt, and the "reduced" generator-end diagram is constructed for $e_g = 1$ volt; and the scales as given in the above table are modified accordingly, but in the "reduced" diagrams P_r, Q_r, P_g and Q_g become p_r, q_r, p_g and q_g as explained in Art. 9.

8. Directions for constructing the "generator end" diagram. This diagram is based on equations (i) and (ii) of Art. 6 exactly as the "receiver-end" diagram is based on equations (iii) and (iv) of Art. 3. Therefore the directions given in Art. 7 can be used for the construction of the "generator end" diagram if we use e_g for e_r, d for a, a for d, δ for α, α for δ, and if we use $\beta \pm 180°$ for β and $\gamma \pm 180°$ for γ.

Lay down values of P_g along the axis $F_g P_g$ in Fig. 12, and lay down values of leading and lagging generator volt-amperes

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along F_gQ_g and F_gQ_g' respectively. Then calculate the P_g and Q_g co-ordinates of the points O and O_g' and locate these points. The circle diagram is then essentially complete.

$$\left.\begin{aligned}
P_g\text{-co-ordinate of } O &= +\frac{d}{b}\cdot e_g^2 \cos(\beta - \delta) \text{ "watts"} \\
Q_g\text{-co-ordinate of } O &= -\frac{d}{b}\cdot e_g^2 \sin(\beta - \delta) \text{ "watts"} \\
P_g\text{-co-ordinate of } O_g' &= +\frac{c}{a}\cdot e_g^2 \cos(\gamma - \alpha) \text{ "watts"} \\
Q_g\text{-co-ordinate of } O_g' &= +\frac{c}{a}\cdot e_g^2 \sin(\gamma - \alpha) \text{ "watts"}
\end{aligned}\right\} \quad \ldots \text{(i)}$$

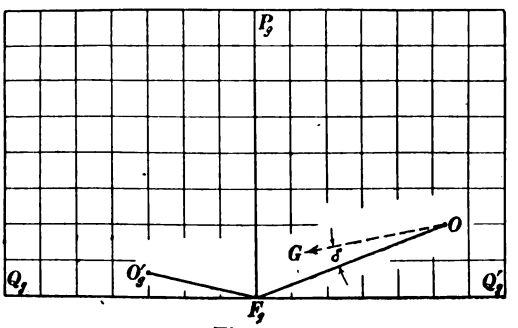

Fig. 12.

"Generator end" diagram. A positive value of P_g (upwards) means power delivered by the generator. Leading volt-amperes is measured towards the left.

9. The "reduced" circle diagram. Suppose a transmission system to be operating steadily with definite values (root-of-mean-square values, of course) of all voltages and currents and with definite constant phase differences between the voltages and currents. *Then there is another steady state of the system for which all voltages and currents are multiplied by n, where n is any numerical multiplier, and for which all phase differences are the same as before.* This proposition is easily established on the basis of the principle of superposition, but the proof need not be given here. Note that if voltages and currents are multiplied by n, with phase differences unchanged, then P and Q are multiplied by n^2. Therefore :—

(*a*) If we construct a "reduced" receiver-end diagram for

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$E_r = 1$ volt, if we use p_r and q_r for power delivered to receiver and wattless volt-amperes at the receiver, respectively, and if we express lengths OM_r, $O'M_r$, etc., in co-ordinate units (watts of p_r), then, *for any given value of E_r equal to $n \times 1$* volt, we have

Scale to which OM_r represents E_g is bn volts per "watt."
,, ,, $O_r'M_r$,, I_g is dn amperes per "watt."
,, ,, F_rM_r ,, I_r is n amperes per "watt."
p_r-co-ordinate of M_r must be multiplied by n^2 to get P_r
q_r-co-ordinate of M_r ,, ,, ,, n^2 ,, Q_r

Note that in placing e_r equal to unity we suppress volts in the denominators of the correct scale numbers b/e_r, d/e_r and $1/e_r$ in the table in Art. 7 so that the precisely correct scales for E_g, I_g and I_r are here (bn/volts) volts-per-"watt," (dn/volts) amperes-per-"watt," and (1/volts) amperes-per-"watt" respectively; but the suppression of volts in the denominators of these scale numbers need lead to no confusion. This note applies also to the next paragraph (*b*).

(*b*) If we construct a "reduced" generator-end diagram for $E_g = 1$ volt, if we use p_g and q_g for power delivered by generator and wattless volt-amperes at generator, respectively, and if we express lengths OM_g, $O'M_g$, etc., in co-ordinate units (watts of p_g), then, *for any given value of E_g equal to $m \times 1$ volt*, we have

Scale to which OM_g represents E_r is bm volts per "watt."
,, ,, $O_g'M_g$,, I_r is am amperes per "watt."
,, ,, F_gM_g ,, I_g is m amperes per "watt."
p_g-co-ordinate of M_g must be multiplied by m^2 to give P_g.
q_g-co-ordinate of M_g ,, ,, ,, m^2 ,, Q_g

Remark. The reduced diagrams differ from Figs. 11 and 12 in scales only, and therefore Figs. 11 and 12 may be thought of as reduced diagrams; but, of course, the co-ordinates must be p_r, q_r, p_g and q_g instead of P_r, Q_r, P_g and Q_g as shown in Figs. 11 and 12.

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10. The Bush and Booth diagram.* It is highly desirable to combine Figs. 11 and 12 in one diagram so that *corresponding* operating points M_r and M_g can be easily located, corresponding points being points M_r and M_g for precisely the same state of operation of the system. To make this combination diagram in an easily usable form it is desirable to reverse the direction of the q-axis in either Fig. 11 or Fig. 12. Let us reverse this axis in Fig. 11, and by so doing we get Fig. 13.

Let us now join Figs. 12 and 13, making the points O coincident and making the lines OG and OR directly opposite to

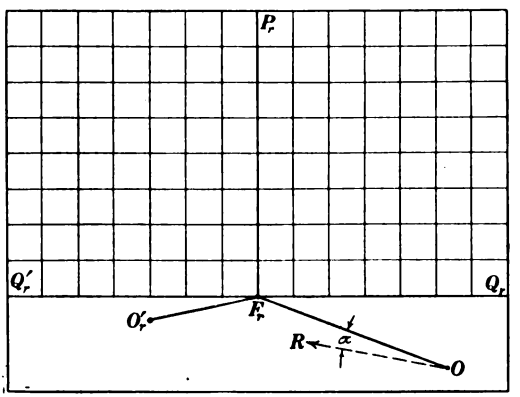

Fig. 13.
Reduced receiver-end diagram with q_r axis reversed. Leading volt-amperes is measured towards the right.

each other as shown in Fig. 14. This is the Bush and Booth diagram.

Proposition I. Corresponding points M_r and M_g in Fig. 14 lie on a straight line passing through the point O.

Proof. The lines OR and OM_r in the receiver-end diagram show the phases of E_r and E_g respectively (see Figs. 7 and 9 for significance of OR); and the lines OG and OM_g in the generator-end diagram show the phases of E_g and E_r respectively. Note that the reversal of the q-axis in Fig. 11 as shown in Fig. 13 has reversed the direction of clock-diagram rotation

* See a paper on Power System Transients, by V. Bush and R. D. Booth, read before the Midwinter Convention of the A.I.E.E., February, 1925.

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so that the curved arrows A and B in Fig. 14 show the respective directions in which an angle of advance in phase is to be measured in the receiver-end and generator-end diagrams which are brought together in Fig. 14. Now Fig. 14 shows $OM_r(= E_g)$ ϕ degrees ahead of $OR(= E_r)$, so that $OM_g(= E_r)$ must be ϕ degrees behind $OG(=E_g)$ as shown, and therefore M_rOM_g is a straight line.

Proposition II. Let the lengths of OM_r and OM_g in Fig. 14 be expressed in co-ordinate units ("watts" of p or q), then OM_r represents E_g to the scale bn volts per "watt" where n is the numerical value of E_r, and OM_g represents E_r to scale bm volts per "watt" where m is the numerical value of E_g (note that reciprocal volts have been suppressed in both of these scale numbers as explained in Art. 9). Therefore we have

$$\frac{bn}{\text{volt}} \cdot OM_r = E_g = m \times 1 \text{ volt}$$

$$\frac{bm}{\text{volt}} \cdot OM_g = E_r = n \times 1 \text{ volt}$$

whence, multiplying, member by member, and cancelling mn, we get

$$b^2 \cdot OM_r \cdot OM_g = 1 \text{ volt}^4 \quad \ldots\ldots\ldots\ldots\ldots \text{(i)}$$

from which the numerical value of OM_g can be calculated when OM_r is known, or *vice versa*. This gives OM_g (or OM_r) in "watts."

11. Construction of the Bush and Booth diagram. To construct this diagram it is necessary to construct the reduced receiver-end diagram (Fig. 13) and the reduced generator-end diagram (Fig. 12) and paste them properly on a large sheet of paper. Of course the co-ordinates are p_r, q_r, p_g and q_g instead of P_r, Q_r, P_g and Q_g.

To choose the proper range of values of p_r and q_r so that the scales of p_r and q_r can be laid down, note that the approximate value of E_r is known ($=n \times 1$ volt) and that the approximate maximum value of P_r is known since it will have been decided to deliver a certain amount of power at a specified voltage.

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Therefore the maximum p_r $(=P_r/n^2)$ is known, and the maximum value of q_r may be safely taken as $\frac{1}{2}$ or $\frac{2}{3}$ of p_r.

Scales of p_r, q_r, p_g and q_g must be the same.

Calculate the p_r and q_r co-ordinates of O in Fig. 13, using equations (i) of Art. 7, taking $e_r = 1$ volt, and remembering that positive values of q_r (leading volt-amperes) are measured towards the right in Fig. 13.

Calculate the p_g and q_g co-ordinates of O in Fig. 12, using equations (i) of Art. 8, and taking $e_g = 1$ volt.

Note that the points O' in Figs. 12 and 13 may be omitted

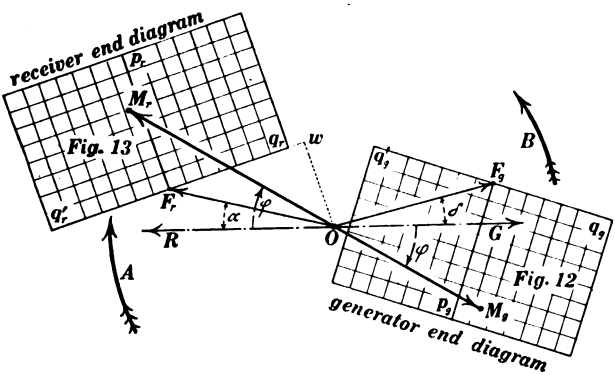

Fig. 14.

The Bush and Booth diagram. Leading wattless volt-amperes are measured from F_r towards q_r and from F_g towards q_g.

without impairing the usefulness of the Bush and Booth diagram.

Paste Fig. 12 on a large sheet of paper as shown in Fig. 14, and lay down the point O and the line OG by laying off the known angle δ.

Draw the line OR opposite to OG and lay off the angle $ROF_r = \alpha$ which is known. Then the line OF_r, whose length is the square root of the sum of the squares of the p_r and q_r co-ordinates of O, can be laid down.

With F_r as a center describe a circle whose radius in the q_r-co-ordinate of O and with O as a center describe a circle

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whose radius is the p_r-co-ordinate of O thus fixing the point w in Fig. 14.

Then paste Fig. 13 on the large sheet of paper as shown in Fig. 14.

12. Use of the Bush and Booth diagram. Let $E_r(=n \times 1$ volt), and the values of P_r and Q_r be given to find all other quantities.

(*a*) Locate the point M_r in Fig. 14 whose p_r and q_r co-ordinates are P_r/n^2 and Q_r/n^2.

(*b*) Measure OM_r (in co-ordinate units), calculate length of OM_g (in co-ordinate units) by using equation (i) of Art. 10, and locate the point M_g in Fig. 14.

Then, all lengths being expressed in co-ordinate units, we have

1. I_r is represented by F_rM_r (see Art. 9).
2. E_g is represented by OM_r (see Art. 9).
3. I_g is represented by F_gM_g (see Art. 9).
4. P_g is obtained by multiplying the p_g-co-ordinate of M_g by m^2.
5. Q_g is obtained by multiplying the q_g-co-ordinate of M_g by m^2, where m is the numeric of E_g as found in 2.

Remark 1. It is evident that there is no need of the points O_r' and O_g' in the Bush and Booth diagram. The phase angles θ_g and θ_r, if they are desired, are easily calculated from the values of P_g, Q_g, P_r and Q_r.

Remark 2. The use of the Bush and Booth diagram, Fig. 14, is greatly facilitated by drawing circles (R-circles, let us call them) in the receiver-end diagram for a series of constant values of $E_g/E_r = 0{\cdot}8, 0{\cdot}9, 1{\cdot}0, 1{\cdot}1, 1{\cdot}2$, etc., and by drawing circles (G-circles, let us call them) in the generator-end diagram for a series of constant values of $E_r/E_g = 0{\cdot}8, 0{\cdot}9, 1{\cdot}0, 1{\cdot}1, 1{\cdot}2$, etc. These circles all center at the point O, and the radius in co-ordinate units is $r_R = (E_g/E_r)/b$ for the R-circles, and $r_G = (E_r/E_g)/b$ for the G-circles. These circles should be labeled with the values of E_g/E_r and E_r/E_g respectively, and they enable the point M_g to be easily located when M_r is given or *vice versa*. Thus if M_r is given the value of E_g/E_r may be read

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off by interpolating between the R-circles, then, of course, E_r/E_g is known, and a line drawn from M_r through O is extended to an ending (the point M_g) whose reading between the G-circles is the value of E_r/E_g.

Remark 3. The calculation of Transmission System Transients by Bush and Booth (as referred to in footnote to Art. 10) involved the calculation of E_g, P_g, Q_g, etc., for many hundreds of sets of given values of E_r, P_r and Q_r, and this work justified the use of an automatic reciprocal linkage device for locating M_g when M_r was given.

PROBLEMS

1. Given a three-wire three-phase line which is to deliver 38,970 kilowatts to three similar receiving circuits at $\triangle$-voltages of 259,700 volts, power factor (lagging) of each receiving circuit being 0·8660. The line wires are arranged as shown in Fig. 1; $D = 6$ feet, $R =$ 0·230 inch. The wires are of aluminum. Taking specific resistance as stated in problem 1 of Chapter III and making allowance for skin effect by multiplying d.c. resistance by 1·009, calculate R, L, and C for the fictitious two-wire line of Fig. 2, and calculate E_r and I_r as shown in Fig. 2. Ans. $R = 0{\cdot}4291$ ohm per mile, $L = 0{\cdot}00193$ henry per mile, $C = 1{\cdot}558 \times 10^{-8}$ farad per mile, $E_r = 150{,}000$ volts, $I_r = 100$ amperes, and I_r lags 30° behind E_r.

2. Using equations (ix), (x), (xi) and (xii) of Art. 3, calculate the values of a, b, c, d, α, β, γ and δ (as defined by equations (v), (vi), (vii) and (viii) of Art. 3) for the 200-mile line which is specified in problem 1 of Chapter III. Use Kennelly's charts to find the polar forms of cosh Ql and sinh Ql and calculate the above quantities by the method explained in the statement immediately following equations (ix), (x), (xi) and (xii) of Art. 3. Ans. $a = 0{\cdot}914$, $b =$ 329 ohms, $c = 0{\cdot}000572$ reciprocal ohms, $d = a$, $\alpha = 3{\cdot}1°$, $\beta =$ $60{\cdot}5°$, $\gamma = 91{\cdot}1°$, $\delta = \alpha$.

Note. The calculation of these results, using the Kennelly charts, took about 10 minutes.

3. Using the values of α^*, β^*, u *and* ψ as given in the answers to

* The symbols α and β are here used each with two different meanings; with asterisks they refer to the real and imaginary components of Q. See Art. 1 of Chapter III.

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problems 2 and 3 of Chapter III, calculate the real and imaginary components of A, B and C using equations (xiii), (xiv), (xv) and (xvi) of Art. 3, and calculate the values of a, b, c, α, β and γ of equations (v), (vi) and (vii) of Art. 3.

Answers :

Real components.	Imaginary components.
$A + 0{\cdot}9165$	$+ 0{\cdot}0492$
$B + 162{\cdot}4$ ohms	$+ 285{\cdot}6$ ohms
$C - 9{\cdot}10 \times 10^{-6}$ reciprocal ohms	$+ 5{\cdot}707 \times 10^{-6}$ reciprocal ohms.

$a = 0{\cdot}9173$, $b = 328{\cdot}6$ ohms, $c = 5{\cdot}709 \times 10^{-4}$ reciprocal ohms,
$\alpha = 3{\cdot}02°$, $\beta = 60{\cdot}38°$, $\gamma = 90{\cdot}90°$.

Note. The calculation of these results took about 50 minutes. The discrepancies between these results and the answers to problem 2 are due mainly to the lack of precision in the use of the Kennelly charts.

4. Using the methods of Art. 7, calculate the P_r and Q_r co-ordinates of the points O and O' in the circle diagram for the line of problem 1, Chapter III for $E_r = 150{,}000$ volts, draw the diagram, locate the point M for $I_r = 100$ amperes and $\theta_r = -30°$ (I_r lagging behind E_r), read off the lengths of the vectors OM and $O'M$ thus getting E_g and I_g, and compare the values of E_g and I_g so found with the values as calculated in problems 3 and 4 of Chapter III.

Answers :

P — co-ordinate of O $= -33{,}880$ "kilowatts."
Q_r — co-ordinate of O $= +52{,}900$ "kilowatts."
P_r — co-ordinate of O' $= -\ \ 513$ "kilowatts."
Q_r — co-ordinate of O' $= -13{,}990$ "kilowatts."
Value of E_g as found from circle diagram $= 167{,}100$ volts.
Value of I_g as found from circle diagram $= 91{\cdot}74$ amperes.

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CHAPTER V

ARTIFICIAL LINES AND WAVE FILTERS

1. The artificial line. A ladder-like arrangement of "series" impedances Z_1 and of "shunt" impedances Z_2 as shown in Fig. 1 is called an *artificial line* because by properly choosing the values of Z_1 and Z_2 this arrangement can be made exactly equivalent to any given transmission line for one particular frequency.*

The subdivision of an artificial line into T-sections. By dividing each series impedance Z_1 into two equal parts (each equal to $\frac{1}{2}Z_1$) Fig. 1 becomes Fig. 2, and the elements or

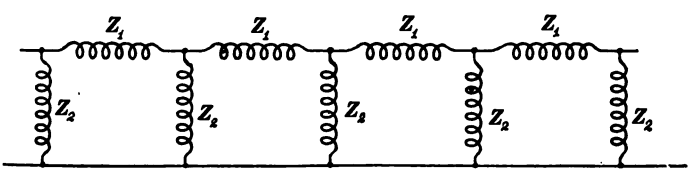

Fig. 1.

parts of Fig. 2 are called *T-sections* because of their resemblance to the letter T.

The subdivision of an artificial line into π-sections. By dividing each shunt impedance Z_2 into two equal parts (each equal to $2Z_2$) Fig. 1 becomes Fig. 3, and the elements or parts of Fig. 3 are called *π-sections* because of their resemblance to the Greek letter π.

Remark 1. The combination of T-sections in Fig. 2 is precisely the same thing as the combination of π-sections in

* An artificial line which can be made to be very nearly equivalent to a given transmission line for all frequencies is described in a paper by F. S. Dellenbaugh; see *Trans.* A.I.E.E., 1923, page 803.

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Fig. 3, *except at the ends of the combination*; a number of complete T-sections cannot terminate like a number of complete π-sections. Note that the points p_1, p_2, etc., in Fig. 2 are covered up, or hidden, in the middles of the series impedances in Fig. 1, and that the points r_1, r_2, etc., in Fig. 3 are covered up, or hidden, in Fig. 1.

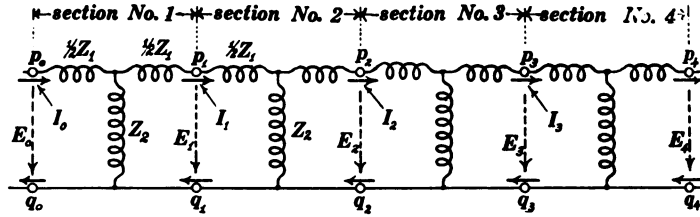

Fig. 2.

A combination of four complete T-sections.

Remark 2. A single T-section or a single π-section can be made exactly equivalent to any given transmission line (for one particular frequency), or any number of T-sections (or π-sections) joined together can be made exactly equivalent to any given transmission line (for one particular frequency).

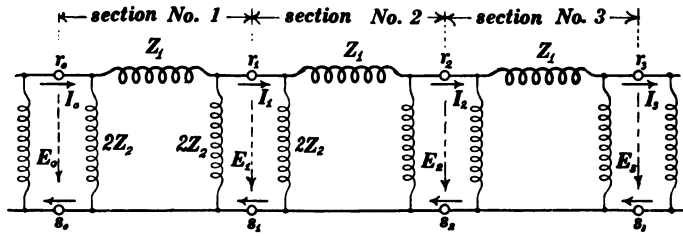

Fig. 3.

A combination of three complete π-sections.

In the latter case the voltage and current vectors E and I at the junctions of the sections are exactly the voltage and current vectors at corresponding points along the given transmission line.

Remark 3. In speaking of voltage and current values along an artificial line *it is customary to refer to the voltage across and*

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the current in the network at the points where the sections are joined together as represented by the letters E_1, $E_2 \ldots I_1$, $I_2 \ldots$ etc., in Figs. 2 and 3. The ratio E_n/I_n has a definite value, if we adopt a point of view which is developed later * and this definite value of E_n/I_n is called the *characteristic impedance* of the artificial line; but this characteristic impedance differs according as we look upon a given arrangement like Fig. 1 as being made up of T-sections or π-sections because E_n and I_n represent voltage and current at *different places in the network* according as we think of the network as being made up of T-sections or π-sections. This matter is easily understood by an inspection of Figs. 1, 2 and 3.

2. The wave filter. The arrangement shown in Fig. 1 (which is reduced to T-sections in Fig. 2 and to π-sections in Fig. 3) is called a *wave filter* when it is designed to transmit well at certain frequencies and to transmit very poorly at other frequencies. These two groups of frequencies are called the *pass frequencies* and the *stop frequencies*, respectively, of the filter. In the wave filter each of the impedances Z_1 and Z_2 is made up of a combination of inductances and condensers.

3. Voltage and current relations along an artificial line or wave filter. We are concerned only with the steady state of the artificial line (where voltage and current have precisely the same frequency everywhere, where phase differences do not change, and where root-of-mean-square values of voltage and current are everywhere constant) and it is easy to show† that the clock-diagram models of Figs. 2 and 3 of Chapter III represent steady states of an artificial line which is built up of any number of similar sections. We are chiefly interested in the steady state which is represented by Fig. 2 of Chapter III and we will therefore discuss this case in some detail.

* The point of view in which we think of a decaying wave-train as traveling along the artificial line.

† It is sufficient to show that Kirchhoff's laws *around a sample mesh* and *at a sample junction point* in Fig. 2 or Fig. 3 are satisfied by the current and voltage vectors in Fig. 2 or Fig. 3 of Chapter III.

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When Fig. 2 of Chapter III represents a steady state of a real transmission line the voltage and current vectors are infinite in number, a voltage vector and a current vector for each point along the line ; but when this clock-diagram model represents a steady state of an artificial line the voltage and current vectors are finite in number, a voltage vector and a current vector for each junction of successive sections of the artificial line. It is evident, therefore, that the steady state of an artificial line as represented by Fig. 2 of Chapter III is not, strictly speaking, a decaying wave-train, but we will use this term nevertheless.

Each section of the artificial line will be thought of as equivalent to *unit length* of a real line so that the angle βl which is mentioned in Art. 7 of Chapter III becomes β radians for each section of the artificial line, and the factor $\epsilon^{\alpha l}$ which is the ratio of decrease in lengths of voltage and current vectors over length l of the real line becomes ϵ^{α} for each section of the artificial line. That is to say, the voltage (and current) vector at section junction n is $n\beta$ radians behind and $1/\epsilon^{n\alpha}$ as long as the voltage vector at section junction O.

From these statements it is evident that

$$E_n = E_0\epsilon^{-(\alpha+j\beta)n} \quad \text{.....(i)}$$

and that

$$I_n = I_0\epsilon^{-(\alpha+j\beta)n} \quad \text{.....(ii)}$$

where E_0 and I_0 are the voltage and current vectors at the left-hand end of Fig. 2 or Fig. 3 and E_n and I_n are the voltage and current vectors at the nth junction towards the right. Thus E_3 is the vector voltage across the third junction and I_3 is the vector current at the third junction as shown in Figs. 2 and 3.

The proof that the clock-diagram model of Fig. 2, Chapter III represents a steady state of an artificial line built up of similar sections, if carried out as above suggested, shows that the ratio E_n/I_n is a constant, a complex constant, so that we may write

$$\frac{E_n}{I_n} = U \doteq u\underline{/\psi} \quad \text{.....(iii)}$$

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The complex quantity U is, of course, an impedance, and it is called the *characteristic impedance* of the artificial line. It is analogous to what we have called the "surge" impedance of a real line. The voltage E_n is numerically equal to uI_n and it is ψ radians ahead of I_n in phase. Of course ψ may be positive or negative.

4. Expressions for α, β and U in terms of Z_1 and Z_2 for a T-section. In Fig. 2 E_0 and I_0 represent voltage and current vectors of a decaying wave-train entering T-section No. 1, and E_1 and I_1 represent voltage and current vectors at the next section junction. Figure 4 shows an enlarged

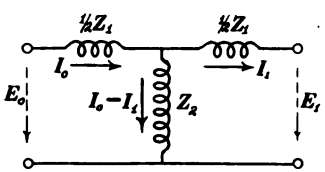

Fig. 4.
Enlarged view of section No. 1 of Fig. 2.

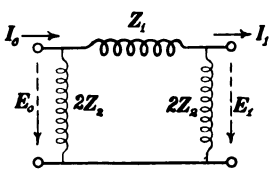

Fig. 5.
Enlarged view of section No. 1 of Fig. 3.

view of T-section No. 1 of Fig. 2, and from Fig. 4 it is easily shown that

$$E_0 = I_0(\tfrac{1}{2}Z_1 + Z_2) - I_1Z_2 \quad \text{.....(i)}$$

and that

$$E_1 = I_0Z_2 - I_1(\tfrac{1}{2}Z_1 + Z_2) \quad \text{.....(ii)}$$

also, since both E_0/I_0 and E_1/I_1 are equal to U, we have

$$E_0/I_0 = E_1/I_1 \quad \text{.....(iii)}$$

whence, eliminating I_0 and I_1 from equations (i), (ii) and (iii) and remembering that $E_1 = E_0\epsilon^{-(\alpha+j\beta)}$, we get

$$\cosh(\alpha + j\beta) = \tfrac{1}{2}\left[\epsilon^{+(\alpha+j\beta)} + \epsilon^{-(\alpha+j\beta)}\right] = 1 + Z_1/(2Z_2) \quad \text{(iv)}$$

which, being a complex-quantity equation, gives the values of both α and β in terms of Z_1 and Z_2.

Let U_T represent the value of the characteristic impedance U of an artificial line built of T-sections. Then $E_1 = U_TI_1$ and from equation (ii) we get

$$U_TI_1 = I_0Z_2 - I_1(\tfrac{1}{2}Z_1 + Z_2) \quad \text{.....(v)}$$

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whence, using $I_1 = I_0\epsilon^{-(\alpha+j\beta)}$ and eliminating $\epsilon^{-(\alpha+j\beta)}$ between the resulting equation and equation (iv), we get

$$U_T = \sqrt{Z_1Z_2 + \tfrac{1}{4}Z_1^2} \quad \ldots\ldots\ldots\ldots\ldots \text{(vi)}$$

which gives the value of U_T in terms of Z_1 and Z_2.

5. Expressions for α, β and U in terms of Z_1 and Z_2 for a π-section. Figure 5 is an enlarged view of section No. 1 of Fig. 3 and by proceeding as in Art. 4 we get

$$\cosh(\alpha + j\beta) = \tfrac{1}{2}[\epsilon^{+(\alpha+j\beta)} + \epsilon^{-(\alpha+j\beta)}] = 1 + Z_1/(2Z_2) \quad \text{(i)}$$

and

$$U_\pi = \frac{Z_1Z_2}{\sqrt{Z_1Z_2 + \tfrac{1}{4}Z_1^2}} = \frac{Z_1Z_2}{U_T} \quad \ldots\ldots\ldots \text{(ii)}$$

Remark 1. Equation (i) of this article is identical to equation (iv) of Art. 4. *Equations (iv) and (vi) of Art. 4 and equations (i) and (ii) of this article are of fundamental importance in the theory of the artificial line and of the wave filter.* By using these equations one can calculate the impedances Z_1 and Z_2 which must be assembled in T-sections or π-sections to give an artificial line which is equivalent (at a given frequency) to any given real line. It must be remembered that α and β here refer to one section, and if a section is to represent a portion of a real line of length l then $\alpha = \alpha' l$ and $\beta = \beta' l$, where α' and β' are the quantities which are represented by α and β in Chapter III.

Also equations (iv) and (vi) of Art. 4 and equations (i) and (ii) of this article are important because they give the values of α, β and U in terms of Z_1 and Z_2.

Remark 2. We have thus far considered only the steady state of an artificial line which corresponds to Fig. 2 of Chapter III. To get the steady state which corresponds to Fig. 3 of Chapter III (a decaying wave-train traveling towards the left in Figs. 1, 2 and 3 of this chapter) it is only necessary to put $E_n/I_n = -U = u\underline{/180^\circ + \psi}$ and to number the sections in Figs. 2 and 3 towards the left so that E_0 and I_0 will be voltage and current vectors at the right-hand ends of Figs. 2 and 3. The values of α and β are the same as before,

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they are not even reversed in sign when the scheme of numbering the sections is altered as stated. In general both steady states exist together, but a full discussion of the general solution of the artificial line or wave filter which depends on a full consideration of both steady states and a consideration of reflections at both ends is not important in the theory of wave filters and so is beyond the scope of this text.

Remark 3. As was stated in Art. 1 the only essential difference between a set of complete T-sections and a set of complete π-sections is in the endings of the artificial line, and if we were to develope the complete theory of reflections at the ends of an artificial line there would be no need whatever to distinguish between a line built up of T-sections and a line built up of π-sections. The usefulness of the distinction between the T-section and the π-section lies wholly in the ease with which two distinct types of filter endings can be taken into account without bothering about details of reflection. Thus adding halves of π-sections to the ends of a chain of complete T-sections is most easily taken account of by using U_π instead of U_T if the load impedance is very nearly equal to U_π but not nearly equal to U_T, and vice versa. See Art. 11.

6. Attenuation of ideal resistanceless artificial lines. Equations (iv) and (vi) of Art. 4 and (i) and (ii) of Art. 5 are the general equations of the artificial line and they apply to any artificial line whatever. For the special case in which Z_1 and Z_2 are pure reactances of the form $Z_1 = jX_1$ and $Z_2 = jX_2$ these general equations are easy to handle because Z_1/Z_2, Z_1Z_2 and Z_1^2 are all real quantities, so that U_T and U_π are either pure reals or pure imaginaries depending on whether the quantity under the radical, namely, $Z_1Z_2 + \frac{1}{4}Z_1^2$, is positive or negative.

To make Z_1 and Z_2 pure reactances the artificial line would have to be made of inductances without resistance and of condensers without losses. Such an ideal line would consume

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no energy in its impedances (pure reactances) and it could very properly be called a *resistanceless line.* This ideal cannot be completely realized, but an artificial line made of well-designed inductance coils and good quality condensers approaches very nearly to the ideal resistanceless line in its characteristics.

The quantity α is called the *attenuation constant* of the artificial line, and the quantity ϵ^{α} is called the *attenuation ratio,* that is to say, the voltage and current vectors are both reduced in numerical value in the ratio of ϵ^{α} to 1 from one junction of two sections to the next junction of two sections (in the direction of travel of the decaying wave-train); and when $\epsilon^{\alpha} = 1$ or $\alpha = 0$ the "decaying" wave-train does not "decay" at all but retains its full initial value of voltage and current as it passes along the artificial line. Now the value of α does become zero for the ideal resistanceless line for certain frequencies and these frequencies are given by the following proposition.

Proposition I. "Decaying" wave-trains of all frequencies for which the impedance (reactance) ratio $Z_1/(4Z_2)$ lies between 0 and -1 do not "decay" at all but traverse the artificial line without diminution of voltage and current. This means that the attenuation constant α is zero for all frequencies which satisfy the condition $0 \geqq Z_1/(4Z_2) \geqq -1$.

To prove this proposition place $\alpha = 0$ in equation (iv) of Art. 4 and we have

$$\tfrac{1}{2}(\epsilon^{j\beta} + \epsilon^{-j\beta}) = 1 + Z_1/(2Z_2) \quad \ldots\ldots\ldots \text{(i)}$$

But $\epsilon^{j\beta}$ is a vector of unit length making an angle $+\beta$ with the axis of reals, and $\epsilon^{-j\beta}$ is a vector of unit length making an angle $-\beta$ with the axis of reals, and we may express these unit vectors in terms of their real and imaginary components so that (i) becomes

$$\tfrac{1}{2}(\cos\beta + j\sin\beta + \cos\beta - j\sin\beta) = 1 + Z_1/(2Z_2)$$

or

$$\cos\beta = 1 + Z_1/(2Z_2) \quad \ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

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Now $\cos \beta$ must lie between $+1$ and -1 and equation (ii) which is true when $\alpha = 0$ shows that $Z_1/(2Z_2)$ must lie between 0 and -2 or that $Z_1/(4Z_2)$ must lie between 0 and -1 thus proving the proposition.

Therefore when the frequency of the voltage which is applied to one end of a resistanceless artificial line is such that the value of $Z_1/(4Z_2)$ lies between 0 and -1, then the voltage sets up a wave-train which traverses the artificial line without attenuation, and such frequencies are called *pass* frequencies because wave-trains of these frequencies pass freely through or over the line. The only effect of the line on wave-trains of pass frequencies is to shift the phase of voltage and current β radians backwards for each section of the line as explained in Art. 3. This phase shift per section is determined by equation (ii), above, which shows that β may have any value from 0° to 360° although, of course, a backward shift of more than 180° may be looked upon as a forward shift. Looking upon the phase shift β as ranging from 0° to $\pm 180°$ it may be stated that β always grows numerically larger for greater and greater frequency within the range of pass frequencies.

Proposition II. When the frequency of the voltage which is applied to one end of an artificial line is such that α is *not* equal to zero [such that $Z_1/(4Z_2)$ does *not* lie between 0 and -1], then the applied voltage sets up a wave-train which decays or is attenuated as it passes along the artificial line and if the line consists of several sections this attenuation may be very great. Such frequencies are accordingly called *stop* frequencies. The degree of attenuation $\epsilon^{n\alpha}$ is determined by the number n of sections and by the value of α, and it can be easily calculated when the value of α is known [as determined by equation (iv) of Art. 4 or by the following equation (iii)]. *It can be shown that for the stop frequencies the angle β is either* $-180°$ *or* 0° *for frequencies less than the pass frequencies and either* 0° *or* $+180°$ *for frequencies greater*

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than the pass frequencies, and for these values of β equation (iv) of Art. 4 reduces to

$$\cosh \alpha = 1 + Z_1/(2Z_2) \quad \ldots\ldots\ldots\ldots \text{(iii)}$$

so that this simplified equation can be used to calculate the attenuation constant α of a resistanceless artificial line.*

7. Making the resistanceless line into an effective wave filter by the use of proper load impedance. Taking such values of Z_1 and Z_2 (pure reactances) as to give a certain range or band of pass frequencies is not sufficient to make the voltages and currents the same at the generator and receiver ends of the artificial line when the frequency of the applied voltage is a pass frequency. This is evident from the following considerations.

A decaying wave-train (which in this case does not decay) travels from the generator end to the receiver end, and this wave-train is in general reflected wholly or in part by the receiver thus producing a reflected wave-train. The voltage in each wave-train is the same in value at both ends of the artificial line (no attenuation) but the phase is shifted $n\beta$ radians backwards in each wave-train where n is the number of sections, and because of this phase shift the vector sum of the voltages (and the vector sum of the currents) in the two wave-trains may be very different in numerical value at the two ends of the line. *The voltages (and currents) can be equal at the two ends of the line only when there is no reflected wave-train.* Then the state of the artificial line consists of a single wave-train traveling from the generator towards the receiver, and, since there is no attenuation, the full voltage (and current) which is delivered to the line by the generator is handed on to the receiver. For frequencies outside of the range of pass frequencies the attenuation is so great that but a small fraction

* A proof of the statements in Proposition II is very similar to the proof of Proposition I and the correctness of the statements relating to the values of β may be easily verified by reference to a table giving the properties of the hyperbolic functions, as for example Appendix B of *Artificial Electric Lines*, by A. E. Kennelly.

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of the voltage (and current) delivered by the generator reaches the receiver end of the line.

In order to eliminate the reflected wave-train (for pass frequencies) the impedance of the receiver must be equal to the characteristic impedance U of the artificial line. When this condition is satisfied the receiver "swallows up" the wave-train which travels out from the generator end of the line. This condition is $U = Z_L$, where Z_L is the impedance of the "load" or receiver, and this condition can be exactly satisfied for only one particular frequency so that if the pass frequencies constitute a fairly wide band the condition can only be approximately satisfied throughout the pass band.

It is not necessary that $U = Z_L$ for the stop frequencies because these frequencies are greatly attenuated by the filter; indeed it is desirable that U be *not* equal to Z_L for the stop frequencies so that the energy of any stop frequency which does reach the receive end of the filter will be largely reflected by the receiver.

An examination of equations (vi) of Art. 4 and (ii) of Art. 5 shows that U_T and U_π are pure resistances for the pass frequencies, and it is desirable therefore that Z_L be a pure resistance and as nearly equal to U_T or U_π as the case may be.

8. Examples of ideal lossless wave filters. The following three examples are the simplest types of wave filters, and a discussion of these three examples shows how the general wave-filter theory works out when applied to specific cases.

The low-pass filter. This wave filter is made with inductances as the series impedances and condensers as the shunt impedances as shown in Fig. 6. For this filter $Z_1 = j\omega L$ and $Z_2 = -j/(\omega C)$, so that $Z_1/(4Z_2) = -\omega^2 LC/4$. Applying proposition I of Art. 6 to this case, we find that the filter shown in Fig. 6 will pass all frequencies below a certain value, and will

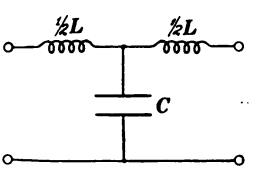

Fig. 6.
Low-pass T-filter section.

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attenuate all frequencies above this value. This filter is accordingly called a *low-pass* filter. The highest frequency it transmits we will call f_2, it is the frequency for which $Z_1/(4Z_2) = -1$, and it is given by the equation

$$f_2 = \frac{1}{\pi\sqrt{LC}}$$

This frequency is called the *cut-off frequency*, since it divides the stop and pass frequencies.

With proper termination the low-pass filter would transmit all frequencies from zero to f_2 without attenuation, and attenuate all higher frequencies even when improperly terminated. The proper termination is that which makes $U_T = Z_L$, and U_T can be found from equation (vi) of Art. 4. Accordingly we find, for the filter being studied,

$$U_T = Z_L = \sqrt{L/C - \omega^2 L^2/4}$$

For frequencies less than f_2 this comes out real, or resistive, as has been already stated. The desired value of load resistance depends on the pass frequency, and varies from 0 to $\sqrt{L/C}$ ohms.

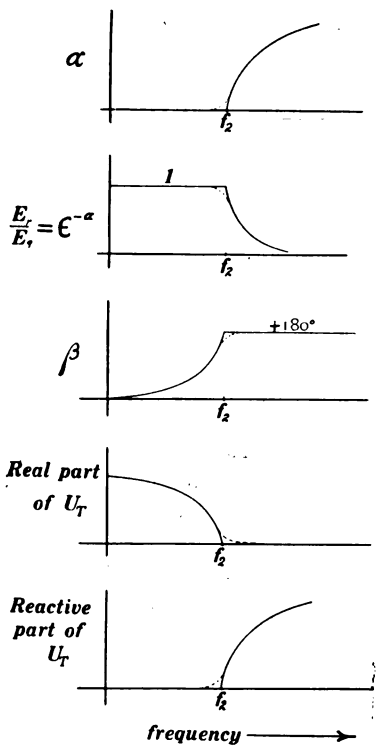

Fig. 7.
Characteristics of low-pass filter.

The angle by which the receiver voltage lags behind the generator voltage is β, and for the pass frequencies it is found by equation (ii) of Art. 6 to be

$$\text{Cos}\ \beta = 1 - \omega^2 LC/2$$

The angle β increases from 0 to 180° as the frequency changes from 0 to f_2. For the stop frequencies $\beta = 180°$, and the

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attenuation constant, as given by equation (iii) of Art. 6, is

$$\text{Cosh } \alpha = 1 - \omega^2 LC/2$$

The detailed characteristics of the filter in Fig. 6 have been computed by the above equations for the case in which $L = 0{\cdot}04$ henry and $C = 4 \times 10^{-6}$ farads, and they are shown in Fig. 7.

High-pass filter. This filter is made with condensers in series and inductances in shunt, as shown in Fig. 8. For this filter $Z_1 = -j/(\omega C)$ and $Z_2 = j\omega L$. Accordingl,y $Z_1/(4Z_2) = -1/(4\omega^2 LC)$ and, since this lies between 0 and -1 for all frequencies which do not make ω smaller than $2\pi f_1$ where f_1 is the cut-off frequency as given below, the filter in Fig. 8 is called a *high-pass filter*. Properly terminated it would transmit all frequencies from infinity down to a frequency f_1 which is given by the equation

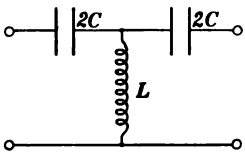

Fig. 8.
High-pass T-filter section.

$$f_1 = \frac{1}{4\pi\sqrt{LC}}$$

This frequency is called the *cut-off frequency* of the high-pass filter, for it marks the transition from pass to attenuated frequencies. Other quantities of interest are $U(=Z_L)$, β, and α, which can be computed from equation (vi) of Art. 4 and equations (ii) and (iii) of Art. 6. Figure 9 shows the detailed characteristics of the Fig. 8 filter computed from the above equations for the

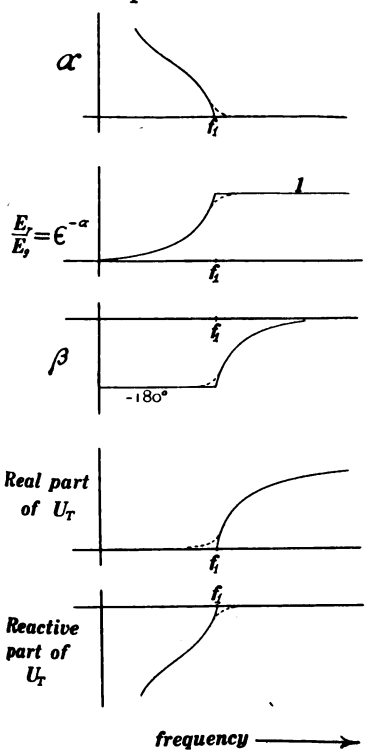

Fig. 9.
Characteristic of high-pass filter.

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special case in which $L = 0{\cdot}04$ henrys and $C = 0{\cdot}25 \times 10^{-6}$ farads.

Band-pass filter. This filter is made up as shown in Fig. 10, and has a value of $Z_1/(4Z_2)$ given by the equation

$$\frac{Z_1}{4Z_2} = \frac{C_2}{4C_1} - \frac{\omega^2 LC_2}{4}$$

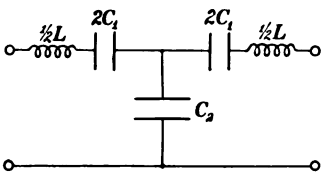

Fig. 10.
Band-pass T-filter section.

When ω is small, the ratio $Z_1/(4Z_2)$ is positive, and when ω is very large the ratio is negative, but less than -1. Between these extremes there is a continuous band of frequencies for which the ratio lies between 0 and -1. This filter is accordingly called a *band-pass* filter. The highest frequency it passes is that which makes $Z_1/(4Z_2) = -1$, and it is

$$f_2 = \frac{1}{2\pi}\sqrt{\frac{C_2 + 4C_1}{LC_1C_2}}$$

while the lowest pass frequency is that for which $Z_1/(4Z_2) = 0$, and it is

$$f_1 = \frac{1}{2\pi\sqrt{LC_1}}$$

These critical frequencies f_1 and f_2 are called the cut-off frequencies. Applied voltages of all frequencies higher than f_2 or lower than f_1 set up waves in the filter which are attenuated in transmission, while applied voltages of frequencies between f_1 and f_2 set up waves that are transmitted without attenuation. When the load impedance equals U, the voltage across the load is the same as the applied voltage for the pass frequencies, as explained above. The quantities α, β and U $(= Z_L)$ can be computed as with the other filters by the use of equation (vi) of Art. 4 and equations (ii) and (iii) of Art. 6. Fig. 11 shows the detailed characteristics of the Fig. 10 filter for the special case in which $L = 0{\cdot}053$ henrys, $C_1 = 0{\cdot}053 \times 10^{-6}$ farads, and $C_2 = 0{\cdot}0707 \times 10^{-6}$ farads.

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Remarks. The three examples that have been considered are all T-section filters, but the results would have been very similar if they were all π-section filters with the same values of Z_1 and Z_2. In fact, the only difference in the characteristics of π and T-section filters having the same Z_1 and Z_2 lies in the characteristic impedance U, and hence in the load impedance required to make the filter function properly. The α and β are the same for both π and T-sections. For the π-section one uses equation (ii) of Art. 5 to obtain U, instead of equation (vi) of Art. 4.

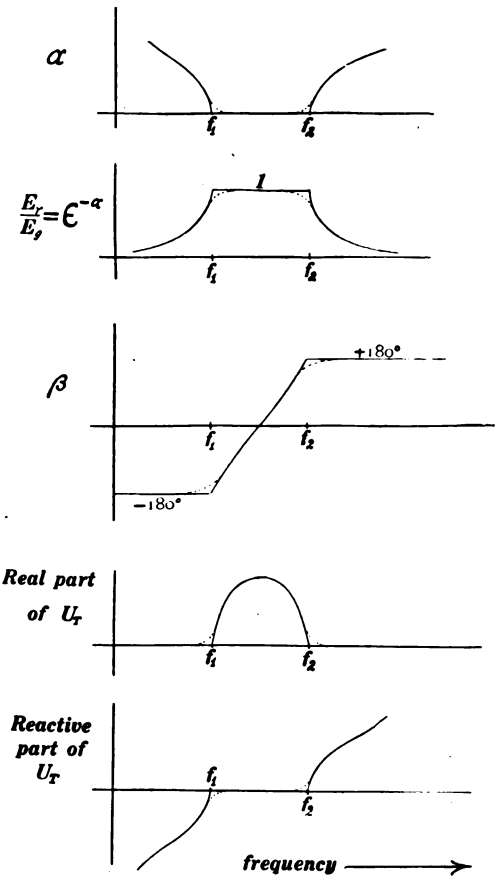

Fig. 11.
Characteristics of band-pass filter.

The examples which have been considered are by no means the only filter sections possible. Some other types are shown in Fig. 12, and while the sections shown in Fig. 12 are in most cases rather complicated, yet the characteristics of these filters are determined by the same equations that apply to the cases discussed above. The only difference lies in the more complicated expressions for Z_1 and Z_2 that are obtained when the filter becomes more complex, but the method of handling Z_1 and Z_2 is the same in all cases.

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9. Effect of losses on filter characteristics. The ideal resistanceless filter discussed in Arts. 6, 7 and 8 can never be fully realized in practice, for while condensers are practically perfect, it is impossible to construct an inductance without appreciable resistance. Fortunately the ratio of reactance to

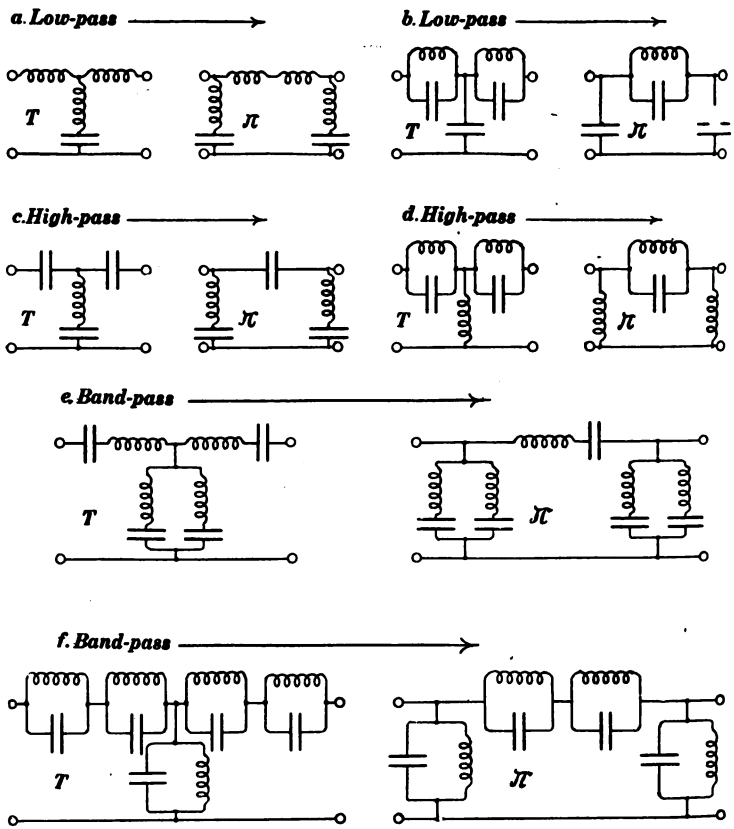

Fig. 12.

resistance of well-built inductances is so high that the effect of the losses is only secondary. The losses slightly modify the characteristics of the ideal filter, but never to an extent to seriously invalidate any of the essential conclusions drawn from a study of the ideal resistanceless filter.

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In Figs. 7, 9 and 11, the dotted curves show the characteristics that would exist if the inductance of the filter had a moderate amount of resistance. In Figs. 7 and 9 this resistance was taken as 10 ohms, while in Fig. 11 it was taken as 12 ohms. These values of resistance are somewhat greater than would be the case if the coils were of the very best design, so the dotted curves in Figs. 7, 9 and 11 give a very good idea of the maximum extent to which the practical case falls below the ideal.

A detailed comparison shows that resistance introduces a small but noticeable attenuation of the wave-trains of pass frequencies, causes the proper value of load impedance to have a very small reactive component for the pass frequencies, and tends to round out the abrupt change of characteristics that marks the transition from pass to stop frequencies in the ideal filter. On the whole, a small amount of loss is not of very great importance.

When computing filter characteristics with the losses taken into account, equations (iv) and (vi) of Art. 4 and equations (i) and (ii) of Art. 5 must be used instead of the simplifications of these equations in Art. 6. It will ordinarily be necessary to use charts of complex hyperbolic functions to determine α and β for both the stop and pass frequencies.

In order to show in detail how to proceed let us study the effect of a resistance R in the inductance of the low pass filter which is shown in Fig. 7. For this filter we then have

$$Z_1 = R + j\omega L$$
$$Z_2 = -j/\omega C$$
$$\text{Cosh}\,(\alpha + j\beta) = 1 + Z_1/(2Z_2) = -\frac{\omega^2 LC}{2} + j.\frac{R\omega C}{2} \quad \text{..(i)}$$
$$U_T = \sqrt{Z_1 Z_2 + Z_1^2/4}$$
$$= \sqrt{\left(\frac{L}{C} - \frac{\omega^2 L}{4}\right) + \frac{R^2}{4} + j\left(\frac{R\omega L}{2} - \frac{R}{\omega C}\right)} \quad \text{.....(ii)}$$

To obtain α and β, find the hyperbolic angle whose cosh is equal to the right-hand member of equation (i), doing this by the use of charts of complex hyperbolic functions as

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explained in Appendix D. The value of U is determined by direct computation by equation (ii).

10. Determination of filter character by inspection. By considering the implications of proposition I in Art. 6 it is possible to determine the approximate location of the stop and pass bands of frequencies of any filter by inspection. The expression $Z_1/(4Z_2)$ will be zero for every frequency which makes $Z_1 = 0$ and for every frequency which makes $Z_2 =$ infinity.* Such frequencies must accordingly lie in a pass band, and this forms an easy method of determining the approximate locations of these bands. Applying this principle to the cases studied in Art. 8, we get $Z_1 = 0$ for zero frequency with the low-pass filter, for infinite frequency in the case of the high-pass filter, and for a frequency which makes $\omega L_1 = 1/(\omega C_1)$ with the band-pass filter.

Similarly, every frequency which makes $Z_1 =$ infinity, and every frequency which makes $Z_2 = 0$ must lie in an attenuating band of frequencies, and this forms a convenient way to locate the stop bands.† Applying this principle to the cases studied in Art. 8, the low-pass filter must have a stop band that includes infinite frequency, the high-pass filter must have a stop band that includes zero frequency, and the band-pass filter must have two stop bands, one including $f = 0$ since zero frequency makes Z_1 infinite, and another including $f =$ infinity, for at infinite frequency $Z_2 = 0$.

This same method of analysis can be applied to any filter, and with a little practice makes it possible for one to determine the general type of even very complicated filters with ease.

11. The steady state solution of a wave filter improperly terminated. When the impedance of the receiver Z_L is equal to the characteristic impedance U of the filter the steady state of the filter consists of one decaying wave-train only,

* Unless $Z_2 = 0$ when $Z_1 = 0$, and $Z_1 =$ infinity when $Z_2 =$ infinity, but these exceptions are ordinarily never encountered.

† Unless both Z_1 and Z_2 become infinite for the same frequency or zero for the same frequency, which is seldom the case.

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and this decaying wave-train is completely "swallowed up" by the receiver (not at all reflected); but when Z_L is not equal to U the steady state of the filter consists of two oppositely traveling wave-trains, and the complete formulation of the state of the filter (the calculation of ratios of generator and receiver voltages and currents and of power delivered by generator and power delivered to receiver) for this case could be developed in a manner essentially identical to the method used in Arts. 9–15 of Chapter III. This general solution is beyond the scope of this text.

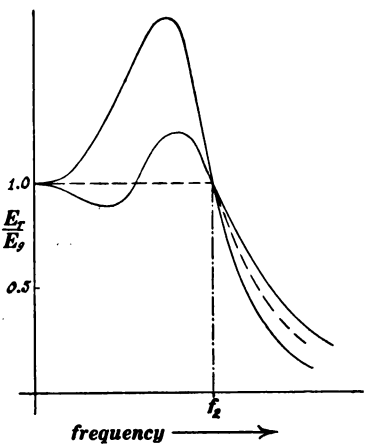

Fig. 13.

The proper value of load impedance is $Z_L = U$; for the lossless filter this is a pure resistance as has been pointed out before; and the value of this resistance varies with the frequency (frequency inside of the pass band, of course) so that it is not possible to satisfy the condition $Z_L = U$ for all pass frequencies simultaneously. When Z_L is equal to U generator and receiver voltages are equal for a particular pass frequency; when Z_L is *not* equal to U generator and receiver voltages are not equal even though the attenuation be zero. This effect is strikingly shown in Fig. 13. The ordinates of the dotted curve show the values of E_r/E_g for different frequencies when the value of Z_L is adjusted so as to be equal to U at each frequency, and the ordinates of the two full-line curves show the values of E_r/E_g for different frequencies for two different constant values of Z_L.

12. Filter sections in series. Uniform filters and composite filters. When a number of similar filter sections are connected in series as shown in Figs. 1, 2 and 3 we have what is called

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a *uniform filter*, when unlike filter sections are connected in series we have what is called a *composite filter*.

When unlike filter sections are connected in series it is very important that the characteristic impedance U be the same for all. When this condition is satisfied a decaying wave-train of any pass frequency will travel through section after section of the filter without being reflected or attenuated, but if U is different for different sections the wave-train will be partially reflected at every junction and only a fraction of the wave-train will reach the load or receiver end of the filter.

A wave-train is *not* reflected in transit when U is the same for all sections, because in this case the voltage and current vectors of the wave-train as it leaves one section have the proper ratio and the proper phase difference to enter the next section. This allows the wave-train to pass from section to section without reflection, or, in other words, each section "swallows up" the wave-train completely as it comes from a preceding section.

The attenuation constant α of a series of sections is

$$\alpha = \alpha_1 + \alpha_2 + \alpha_3 + \text{etc.} \quad \ldots\ldots\ldots\ldots \text{(i)}$$

where α_1, α_2, α_3, etc., are the attenuation constants of the respective sections. This is evident because the first section reduces the voltage (and current) in the wave-train to $E_0\epsilon,^{-\alpha_1}$ and this is reduced by the factor $\epsilon^{-\alpha_2}$ by the second section giving $E_0\epsilon^{-\alpha_1}\epsilon^{-\alpha_2}$ or $E_0\epsilon^{-(\alpha_1+\alpha_2)}$, and so on. The ratio of receiver-end voltage E_r to generator-end voltage E_g of a decaying wave-train traveling towards the receiver is

$$\frac{E_r}{E_g} = \frac{I_r}{I_g} = \epsilon^{-\alpha} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

where α has the value given by equation (i).

13. Composite wave filters. The characteristics of the simple filter sections which are discussed in Art. 8 are unsatisfactory in several respects. (*a*) In order to obtain the best possible transmission of the pass frequencies the load

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resistance would have to be different for different pass frequencies because U is different, and this is impractical; (*b*) The attenuation is by no means the same for all the stop frequencies, stop frequencies that are near to the pass frequencies are, relatively, only slightly attenuated.

A filter with a constant-resistance load will function satisfactorily only if the characteristic impedance U of the filter is approximately constant throughout the pass band, and this is obviously not the case with the filters of Figs. 6, 8 and 10 whose characteristics are shown in Figs. 7, 9 and 11. Fortunately it is possible to transform the U of a filter from one value to another by the use of *half-sections* as explained in Art. 4. Ending the filter with properly designed half-sections makes it possible to transform the undesirable characteristic impedance of the filter into a much more desirable one, one which is practically constant over the greater part of the pass band.

No type of filter section has been discovered which gives a uniformly high attenuation over the entire stop band. The attenuation is always very high for some frequencies and undesirably low for other frequencies. In Fig. 7, for example, the attenuation is low near the cut-off frequency, and high at $f = \infty$. Other low-pass filters have an attenuation that increases to a very high value for frequencies only slightly different from the cut-off frequency, but these filters always have poor attenuation at high frequencies; and so it goes.

To obtain a uniformly high attenuation for all stop frequencies one may use a composite filter built up of unlike filter sections in series. If these unlike sections are chosen so that the frequencies weakly attenuated by one section are strongly attenuated by another section, the combination has a much more desirable characteristic than sections of either kind alone.

Composite filters are practically the only kind used in actual service, because they can be made to have uniformly high attenuation throughout the stop band of frequencies with

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the use of only a small number of sections, and they will function satisfactorily with a constant resistance load when the ends are suitable half sections.

The filter which is built up of similar sections in series (the so-called uniform filter) is not often used because all sections have the same points of weak attenuation and the same points of strong attenuation, so that the uniform filter has a surplus of attenuation for certain frequencies and a deficiency of attenuation for other frequencies.

The advantages of the composite filter are brought out clearly by studying a specific case. In Fig. 14 the solid line

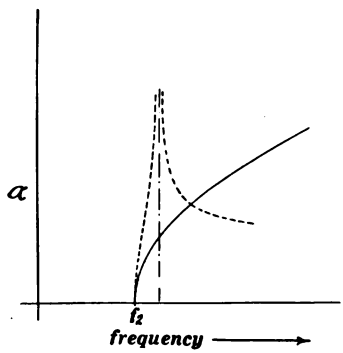

Fig. 14.

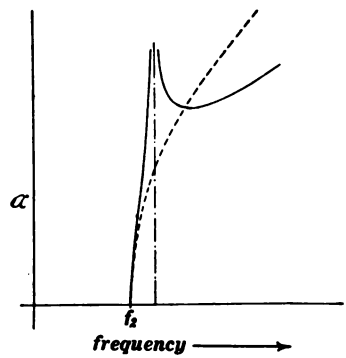

Fig. 15.

represents the attenuation characteristic of a low-pass section of the kind shown in Fig. 6 and the dotted line represents the attenuation characteristic of another type (see Arts. 14 and 15) of low-pass section which has the same pass frequencies and the same U as the first section. When two sections like Fig. 6 are connected in series the attenuation constant of the two is twice that of one section alone as shown by the ordinates of the dotted curve in Fig. 15 (the ordinates of the dotted curve in Fig. 15 are twice the ordinates of the full-line curve in Fig. 14). When one section of each type are connected together the attenuation constant of the combination is the sum of the attenuation constants of the respective sections as shown by the ordinates of the full-line curve in Fig. 15 (the

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ordinates of the full-line curve in Fig. 15 is the sum of the ordinates of the two curves in Fig. 14).

The advantages of the composite filter (full-line curve of Fig. 15) are obvious; the attenuation rises rapidly as soon as the cut-off frequency f_2 is passed and the attenuation is fairly high over the whole range of stop frequencies.

The design of the composite filter depends on the finding of some type of filter section for which the attenuation characteristics can be altered without changing U and without changing the cut-off frequencies. This condition is realized in the *inverse-network filter.*

14. The inverse-network filter. The inverse-network filter is a filter in which the elements Z_1 and Z_2 are so related that the product Z_1Z_2 is a constant which is independent of frequency. The name of this type of filter arises from the fact that Z_1 and Z_2 may be considered "inverse networks" when the product of Z_1Z_2 is constant.

All inverse-network filters have three properties in common. In order that Z_1Z_2 may be a constant, it is necessary that Z_2 become infinite whenever Z_1 is zero, and vice versa, for otherwise the product Z_1Z_2 would not be constant. In simpler language, this is equivalent to saying that in the inverse-network filter Z_2 must be an infinite impedance whenever Z_1 is a zero impedance, and conversely Z_2 must be a zero impedance whenever Z_1 is an infinite impedance.

When an inverse network is arranged in the form of a ladder, as shown in Fig. 1, the values of U_T and U_π are the same for the frequency which makes $Z_1 = 0$. The value of U for this frequency will be called R, for it will be shown later that R is the proper value of load resistance. The value of R is given by the equation $R = \sqrt{Z_1Z_2}$, and by the use of equation (vi) of Art. 4 and equation (ii) of Art. 5, it becomes evident that R is the maximum value that U_T has in the pass band of frequencies, and it is also the minimum value which U_π has in the pass band.

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The inverse network filter must also be a resistanceless filter. Otherwise the product Z_1Z_2 cannot be independent of frequency for all frequencies.

15. Design of inverse-network filters. (*a*) *Low-pass inverse-network filter.* The only possible low-pass inverse-network filter has an inductance as the series element and a condenser as the shunt element. Fig. 6 is an example of such a section. This filter is an inverse network because $Z_1Z_2 = (j\omega L)[1/(j\omega C)] = L/C = R^2$ is constant. The values of U for π and T-sections of this filter are found from equation (vi) of Art. 4 and equation (ii) of Art. 5, and they are

$$U_T = \sqrt{R^2 - \omega^2 LC/4}$$

$$U_\pi = \frac{R^2}{\sqrt{R^2 - \omega^2 LC/4}}$$

The highest pass frequency is f_2, and it is the frequency that makes $Z_1/(4Z_2) = -1$, as discussed in Art. 8.

To determine the values of the inductance and of the condenser capacity which are to be used in the inverse-network low-pass filter it is merely necessary to specify the cut-off frequency f_2 and the value of R. With a known R and a known f_2 we have

$$R^2 = Z_1Z_2 = L_k/C_k$$
$$-\tfrac{1}{4}(2\pi f_2)^2 L_kC_k = -1$$

and from these two equations we get

$$L_k = R/(\pi f_2)$$
$$C_k = 1/(\pi f_2 R)$$

The subscript k is applied to inductances and capacities that compose the inverse-network filter, in which the fundamental requirement is that $Z_1Z_2 =$ constant.

(*b*) *High-pass inverse-network filter.* The only possible high-pass inverse-network filter has a condenser as the series element and an inductance as the shunt element. An example of such a filter is shown in Fig. 8. This filter is an inverse

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network because $Z_1Z_2 = [1/(j\omega C)](j\omega L) = L/C = R^2$ is constant. The values of U for the π and T-sections can be computed by the aid of equations (vi) of Art. 4 and equation (ii) of Art. 5, and the lowest pass frequency f_1 can be found from the fact that it is the frequency which makes $Z_1/(4Z_2) = -1$.

To determine the values of the inductance and of the condenser capacity which are to be used in the inverse-network high-pass filter it is merely necessary to specify the cut-off frequency f_1 and the value of R. When this is done L_k and C_k of the high-pass filter can be determined in a manner similar to that used in the case of the low-pass filter, with the result

$$L_k = \frac{R}{4\pi f_1}$$

$$C_k = \frac{1}{4\pi f_1 R}$$

The subscript k is used to denote capacities and inductances that make up the inverse-network filter, in which the fundamental requirement is that $Z_1Z_2 =$ constant.

(c) *Band-pass inverse-network filter.* The only possible band-pass inverse-network filter has the construction shown in Fig. 16. For this filter

$$Z_1 = j\left(\omega L_{1k} - \frac{1}{\omega C_{1k}}\right)$$

$$Z_2 = j\,\frac{\omega L_{2k}}{1 - \omega^2 L_{2k} C_{2k}}$$

When $L_{1k}C_{1k} = L_{2k}C_{2k}$ the filter becomes an inverse-network filter because then $Z_1Z_2 = L_{2k}/C_{1k} = R^2$. The values of U for the π and T-sections can be computed by the aid of equation (vi) of Art. 4 and equation (ii) of Art. 5. The lowest pass frequency f_1 and the highest pass frequency f_2 are both frequencies which make $Z_1/(4Z_2) = -1$.

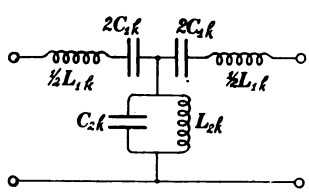

Fig. 16.
T-section of a band-pass filter.

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To determine the values of the two inductances and the two condenser capacities which are to be used in the inverse-network band-pass filter it is sufficient to specify the highest and lowest pass frequencies f_2 and f_1 respectively, and the value of R. This together with the condition $L_{1k}C_{1k} = L_{2k}C_{2k}$ that is given above will determine the four desired quantities with the following result

$$L_{1k} = \frac{R}{\pi(f_2 - f_1)}$$

$$L_{2k} = \frac{(f_2 - f_1)R}{4\pi f_1 f_2}$$

$$C_{1k} = \frac{(f_2 - f_1)}{4\pi f_1 f_2 R}$$

$$C_{2k} = \frac{1}{\pi(f_2 - f_1)R}$$

As before, the subscript k is applied to condensers and inductances that compose the inverse-network filter.

16. Design of resistanceless low-pass composite filters. The most complicated low-pass filters that can be devised have

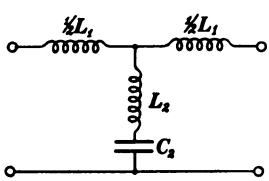

Fig. 17.

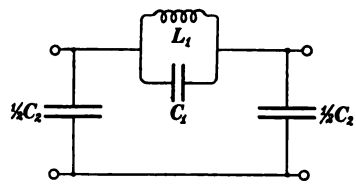

Fig. 18.

three elements, and are of two types. A T-section of one type is shown in Fig. 17, and a π-section of the other possible type is shown in Fig. 18. The attenuation characteristic of both of these filters is of the same general nature, and is shown in Fig. 19. It is seen that there is one frequency in the attenuating band for which the attenuation is infinite. This frequency can be called f_∞. It is the frequency that makes Z_2 become zero for the filter in Fig. 17, and it is the frequency that makes Z_1 become infinity for the filter in Fig. 18. In order to determine the value of f_∞, it is sufficient to note that $Z_2 = 0$ in

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Fig. 17 when $\omega L_2 = 1/(\omega C_2)$, and that $Z_1 =$ infinity in Fig. 18 when $\omega L_1 = 1/(\omega C_1)$.

(*a*) *Design of low-pass composite filters formed of T-sections.* The T filter of Fig. 17 has properties that make it ideally suited for composite filters. Every such T-section designed according to Plate IA for a given R and f_2 has the same characteristic impedance U_T as the low-pass inverse-network T-section built to have this same R and f_2. Since f_2 is the cut-off frequency that determines the pass band of frequencies, and since (as shown in Art. 17) R is the proper value of load resistance, it follows that *all T-sections designed by the equations of Plate IA to operate on a given load resistance R, and to have a given cut-off frequency f_2, have the same characteristic impedance U_T and so can be connected in series.* The attenuating properties of T-sections designed in this way depend upon the location of f_∞ relative to f_2, and Plate IA shows that the value of f_∞ can be chosen at will without varying f_2 or R. The value of the frequency for infinite attenuation, f_∞, determines the proportions of the filter, but this value of f_∞ can be arbitrarily changed without changing R or f_2, and so without changing U_T. This makes it possible to build T-sections which can be put in series, and yet which have widely different attenuating characteristics.

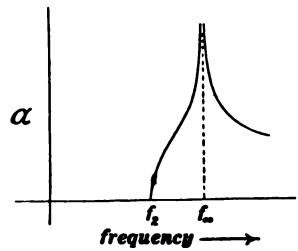

Fig. 19.

That a T-section designed according to Plate IA has the same U_T as the corresponding inverse-network filter can be readily shown. Substituting the values of Z_1 and Z_2 determined by Plate IA into equation (vi) of Art. 4 gives

$$U_T = \left[(j\omega m L_k)\left(j\omega C_k \frac{1 - m^2}{4m} + \frac{1}{j\omega m C_2}\right) - \tfrac{1}{4}(\omega m L_k)^2 \right]^{\frac{1}{2}}$$

Simplifying, this becomes

$$U_T = \sqrt{R^2 - \frac{\omega^2 L_k C_k}{4}}$$

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which is exactly the expression given in Art. 14 for U_T of the low-pass inverse-network filter.

The design given in Plate IA was derived by writing down the equation giving U_T of the T-section of Fig. 17 in terms of L_1, C_2, and L_2, and then determining what values of L_1, L_2, and C_2 would make this characteristic impedance the same as that of the inverse-network filter. The results are given in Plate IA, and they show that it is not necessary to fix f_∞ in order to attain the desired U_T. It is interesting to note that the low-pass inverse-network T filter is the special case of Plate IA in which $f_\infty = \infty$.

(*b*) *Design of low-pass composite filters formed of π-sections.* The π-filter of Fig. 18 has properties that make it ideally suited for composite filters. This is because *all such π-sections designed by the equations of Plate* IB *to operate on a given load resistance R, and to have a given cut-off frequency f_2, have the same characteristic impedance U_π and so can be connected in series.* The attenuating properties of π-sections designed in this way depend on the value of f_∞ in relation to f_2, and this relation can be changed at will without changing f_2 or R, and so without varying U_π.

The method of deriving Plate IB is exactly similar to the method of deriving Plate IA. The proof that U_π of every section designed by Plate IB for a given R and f_2 is the same as the U_π of the corresponding inverse-network π-section is also similar to the proof given in connection with Plate IA. The inverse network π-filter is the special case of Plate IB in which $f_\infty = \infty$.

17. Proper termination of low-pass filters to permit the use of constant-resistance load. To make the filter sections designed according to Plates IA and IB function satisfactorily it would be necessary to have a different load resistance for every frequency in the pass band. This manifestly impractical requirement arises from the fact that these sections have the U_T and U_π of the low-pass inverse-network filter, and these characteristic impedances U_T and U_π of the low-pass inverse-

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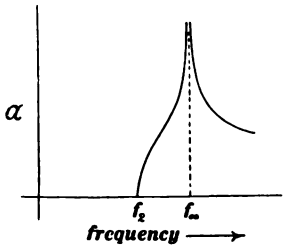
PLATE I

DESIGN OF LOW-PASS SECTIONS

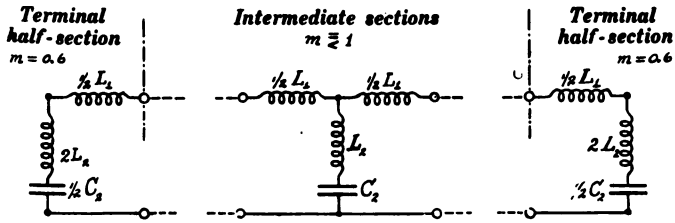


$$L_k = \frac{R}{2f_2}$$

$$C_k = \frac{1}{\pi f_2 R}$$

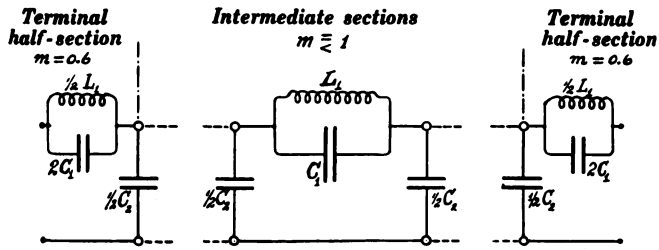
$$m = \sqrt{1 - \left(\frac{f_2}{f_\infty}\right)^2}$$

A. DESIGN OF *T*-SECTIONS



$$L_1 = mL_k \qquad L_2 = \frac{1-m^2}{4m} \cdot L_k \qquad C_2 = mC_k$$

B. DESIGN OF π-SECTIONS



$$L_1 = mL_k \qquad C_1 = \frac{1-m^2}{4m} \cdot C_k \qquad C_2 = mC_k$$

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network filter depend on the frequency which is being transmitted.

In order to partly overcome this disadvantage the filters designed according to Plates IA and IB must be terminated with *half-sections* which transform the characteristic impedance into something more desirable. Investigation has shown that the best way to end a filter made of sections designed according to Plate IA is *to use at each end a half T-section designed for* $m = 0{\cdot}6$. This termination has been indicated in the drawing shown on Plate IA. Investigation has also shown that the best way to end a filter made of sections designed according to Plate IB is *to use at each end a half π-section designed for* $m = 0{\cdot}6$. This termination has been indicated in the drawing shown on Plate IB.

That these terminations are really effective in transforming U from a quantity that changes rapidly with frequency to one that is nearly independent of frequency over the pass bands can be readily shown. The effect of the half T-section in Plate IA is to change the characteristic impedance from $U_T = \sqrt{Z_1 Z_2 + \frac{1}{4} Z_1^2}$ to the value $U_\pi = \dfrac{Z_1 Z_2}{U_T}$. Substituting the values of Z_1 and Z_2 derived from Plate IA, one obtains on simplification

$$\frac{R}{U_\pi} = \frac{\sqrt{1 - (f/f_2)^2}}{1 + (1 - m)^2 \left(\dfrac{f}{f_2}\right)^2}$$

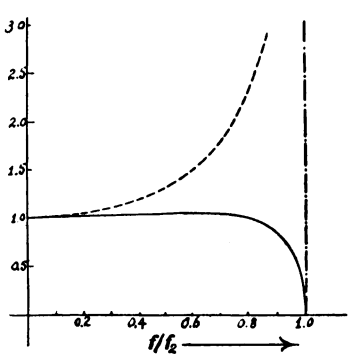

Fig. 20.

Plotting R/U_π against (f/f_2) for $m = 0{\cdot}6$ gives the result shown in Fig. 20, which shows that for $m = 0{\cdot}6$ the ideal load resistance U_π does not differ from the actual constant load resistance R by more than 4% for frequencies covering 85% of the pass band. For the sake of comparison, the ordinates of dotted line in Fig. 20 represent

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the ratio of actual load resistance R to ideal load resistance U_T that the filter would have if not terminated by half sections. The effect of the half π-sections terminating the filter designed by Plate IB is very similar to the result shown in Fig. 20 for the half T-section.

18. Design of resistanceless high-pass composite filters. The most complicated high-pass filters that can be devised have three elements, and are of two types. A T-section of

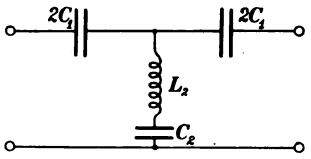

Fig. 21.

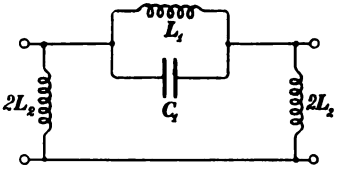

Fig. 22.

one type is shown in Fig. 21, and a π-section of the other possible type is shown in Fig. 22.

The attenuation characteristic of both of these filters is of the same general nature, and is given in Fig. 23. It is seen that there is one frequency in the attenuating band for which the attenuation is infinite. This frequency can be called f_∞. It is the frequency that makes Z_2 become zero for the filter in Fig. 21, and it is the frequency that makes Z_1 become infinity for the filter in Fig. 22.

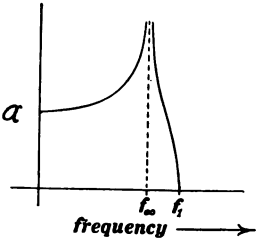

Fig. 23.

(*a*) *Design of high-pass composite filters formed of T-sections.* The T filter of Fig. 21 has properties that make it ideally suited for composite filters. This is because *all such T-sections designed by the equations of Plate* IIA *to operate on a given load resistance R, and to have a given cut-off frequency* f_1 *have the same characteristic impedance* U_T *and so can be connected in series.* The attenuating properties of T-sections designed in this way depend on the value of f_∞ in relation to f_2, and this relation can be changed at will without varying f_2 or R,

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and so without affecting U_T. The inverse-network T-section is a special case of Plate IIA for which $f_\infty = 0$.

(*b*) *Design of high-pass composite filters formed of π-sections.* The π-section of Fig. 22 has properties that make it ideally suited for composite filters. This is because *all such π-sections designed by the equations of Plate* IIB *to operate on a given load resistance* R, *and to have a given cut-off frequency* f_1, *have the same characteristic impedance* U_π, *and so can be connected in series.* The attenuating properties of π-sections designed in this way depend on the value of f_∞ in relation to f_1, and the location of f_∞ can be changed at will without altering f_1 or R, and so without affecting U_π. The high-pass inverse-network π-filter is a special case of Plate IIB for which $f_\infty = \infty$.

19. Proper termination of high-pass filters to permit the use of constant-resistance load. The filters designed according to Plates IIA and IIB will not function satisfactorily with a constant-resistance load, and so must be terminated with half sections designed to change the characteristic impedance into something more desirable. Investigation has shown that the best way to terminate a filter made of sections designed according to Plate IIA is to *use at each end a half T-section designed for* $m = 0{\cdot}6$. This termination has been indicated in the drawing shown on Plate IIA. Investigation has also shown that the best way to terminate a filter made of sections designed according to Plate IIB is to *use at each end a half π-section designed for* $m = 0{\cdot}6$. This termination has been indicated in the drawing shown on Plate IIB.

The effect of such half sections is to keep the ideal load resistance within 4% of constancy for all pass frequencies that differ from the cut-off frequency by more than 15%. This means that the filter equipped with suitable half sections will function almost perfectly on all pass frequencies from $1{\cdot}15f_1$ to infinity. The demonstration of this fact is similar to the demonstration given in Art. 17, and it need not be repeated here.

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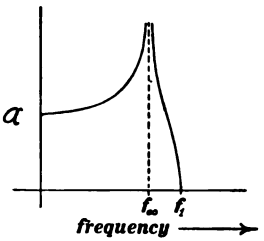
PLATE II

DESIGN OF HIGH-PASS SECTIONS.

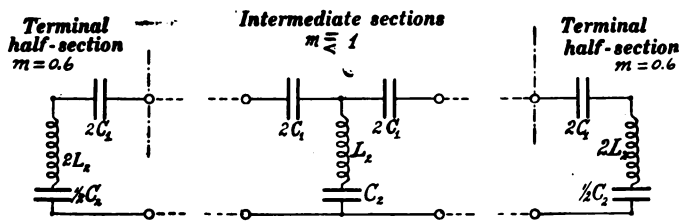


$$L_k = \frac{R}{4\pi f_1}$$

$$C_k = \frac{1}{4\pi f_1 R}$$

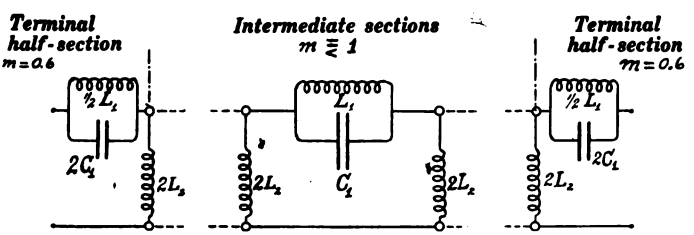
$$m = \sqrt{1 - \left(\frac{f_\infty}{f_1}\right)^2}$$

A. DESIGN OF *T*-SECTIONS



$$C_1 = \frac{C_k}{m} \qquad C_2 = \frac{4m}{1 - m^2} \cdot C_k \qquad L_2 = \frac{L_k}{m}$$

B. DESIGN OF π-SECTIONS



$$L_1 = \frac{4m}{1 - m^2} \cdot L_k \qquad C_1 = \frac{C_k}{m} \qquad L_2 = \frac{L_k}{m}$$

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20. Design of resistanceless band-pass composite filters. The most complicated band-pass filters that can be devised have six elements, and are of two types. A T-section of one type is shown in Fig. 24, and a π-section of the other possible

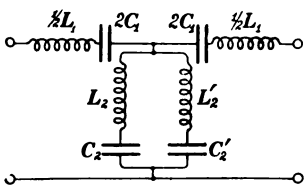

Fig. 24.

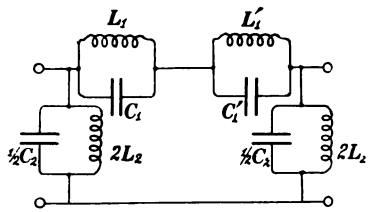

Fig. 25.

type is shown in Fig. 25. The attenuation characteristic of both of these filters is of the same general nature, and is given in Fig. 26.

It is seen that there is one frequency in each attenuating band for which the attenuation is infinite. The lowest frequency of infinite attenuation will be called $f_{1\infty}$ and the highest frequency of infinite attenuation will be called $f_{2\infty}$. There are two frequencies of infinite attenuation in Fig. 24 because this filter has two shunt circuits instead of one as in Figs. 17 and 21, and the filter in Fig. 25 has two frequencies of infinite attenuation because it has two shunt resonant circuits in the series impedance instead of the one in Figs. 18 and 22.

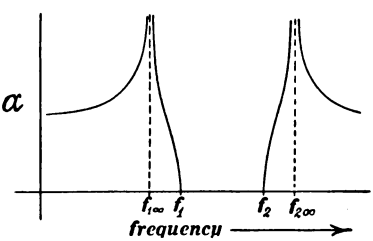

Fig. 26.

(*a*) *Design of band-pass composite filters formed of T-sections.* The T-filter of Fig. 24 has properties that make it ideally suited for composite filters. This is because *all such T-sections designed by the equations of Plate* III A *to operate on a given load resistance R, and to have given cut-off frequencies* f_1 *and* f_2 *have the same characteristic impedance* U_T *and so can be connected in series.* The attenuating properties of T-sections

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designed in this way depend on the values of $f_{1\infty}$ and $f_{2\infty}$, and these frequencies can be given any arbitrary values desired without varying f_1, f_2, or R, and so without affecting U_T. The inverse-network T-section band-pass filter is a special case of Plate IIIA for which $f_{1\infty} = 0$, and $f_{2\infty} = \infty$.

(*b*) *Design of band-pass composite filters formed of π-sections.* The filter of Fig. 25 has properties that make it ideally suited for composite filters. This is because *all such π-sections designed by the equations of Plate* IIIB *to operate on a given load resistance R, and to have given cut-off frequencies f_1 and f_2, have the same characteristic impedance U_π, and so can be connected in series.* The attenuating properties of π-sections designed in this way depend on the values of $f_{1\infty}$ and $f_{2\infty}$, and these frequencies can be given any arbitrary values desired without varying f_1, f_2, or R, and so without affecting U_π. The inverse-network π-section filter is a special case of Plate IIIB for which $f_{1\infty} = 0$ and $f_{2\infty} = 0$.

21. Proper termination of band-pass filters to permit the use of constant-resistance load. The filters designed according to Plates IIIA and IIIB will not function satisfactorily with a constant-resistance load, and so must be terminated with half sections designed to change the characteristic impedance into something more desirable. Investigation has shown that the best way to terminate a filter made of sections designed according to Plate IIIA is to *use at each end a half T-section designed for* $m_1 = m_2 = 0{\cdot}6$. This termination has been indicated in the drawing shown on Plate IIIA. Investigation has also shown that the best way to terminate a filter made of sections designed according to Plate IIIB is to *use at each end a half π-section designed for* $m_1 = m_2 = 0{\cdot}6$. This termination has been indicated in the drawing shown on Plate IIIB.

The effect of the half sections described above is found in a manner similar to that followed in Art. 17, but it need not be gone into at length here. The important fact is that the termination described will keep the ideal load resistance

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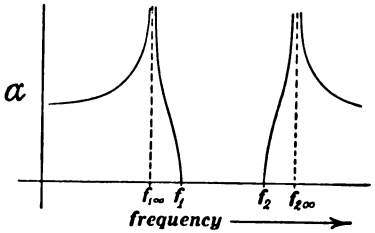
PLATE III

DESIGN OF BAND-PASS SECTIONS



$$L_{1k} = \frac{R}{\pi(f_2 - f_1)} \qquad L_{2k} = \frac{(f_2 - f_1)R}{4\pi f_1 f_2}$$

$$C_{1k} = \frac{f_2 - f_1}{4\pi f_1 f_2 R} \qquad C_{2k} = \frac{1}{\pi(f_2 - f_1)R}$$

$$g = \sqrt{\left(1 - \frac{f_{1\infty}^2}{f_1^2}\right)\left(1 - \frac{f_{1\infty}^2}{f_2^2}\right)} \qquad h = \sqrt{\left(1 - \frac{f_1^2}{f_{2\infty}^2}\right)\left(1 - \frac{f_2^2}{f_{2\infty}^2}\right)}$$

$$m_1 = \frac{\dfrac{f_1 f_2}{f_{2\infty}^2} \cdot g + h}{1 - \dfrac{f_{1\infty}^2}{f_{2\infty}^2}} \qquad m_2 = \frac{g + \dfrac{f_{1\infty}^2}{f_1 f_2} \cdot h}{1 - \dfrac{f_{1\infty}^2}{f_{2\infty}^2}}$$

$$a = \frac{(1 - m_1^2) f_{2\infty}^2}{4 g f_1 f_2} \cdot \left(1 - \frac{f_{1\infty}^2}{f_{2\infty}^2}\right) = \frac{(1 - m_2^2) f_1 f_2}{4 g f_{1\infty}^2} \cdot \left(1 - \frac{f_{1\infty}^2}{f_{2\infty}^2}\right)$$

$$b = \frac{(1 - m_2^2)}{4g} \cdot \left(1 - \frac{f_{1\infty}^2}{f_{2\infty}^2}\right) \qquad c = \frac{(1 - m_1^2)}{4h} \cdot \left(1 - \frac{f_{1\infty}^2}{f_{2\infty}^2}\right)$$

$$d = \frac{(1 - m_1^2) f_{2\infty}^2}{4 h f_1 f_2} \cdot \left(1 - \frac{f_{1\infty}^2}{f_{2\infty}^2}\right) = \frac{(1 - m_2^2) f_1 f_2}{4 h f_{1\infty}^2} \cdot \left(1 - \frac{f_{1\infty}^2}{f_{2\infty}^2}\right)$$

When $m_1 = m_2$ then we have $g = h$, $a = d$, $b = c$,

$$m_1 = m_2 = \frac{h}{1 - f_1 f_2 / f_{2\infty}^2} , \qquad f_{1\infty} = f_1 f_2 / f_{2\infty} \quad \text{and}$$

$$f_{2\infty}^2 = \frac{f_1^2 + f_2^2 - 2m^2 f_1 f_2}{2(1 - m^2)} + \left[\left(\frac{f_1^2 + f_2^2 - 2m^2 f_1 f_2}{2(1 - m^2)}\right)^2 - f_1^2 f_2^2\right]^{\frac{1}{2}}$$

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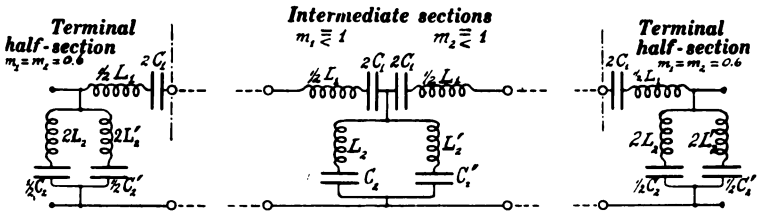
PLATE III

(continued)

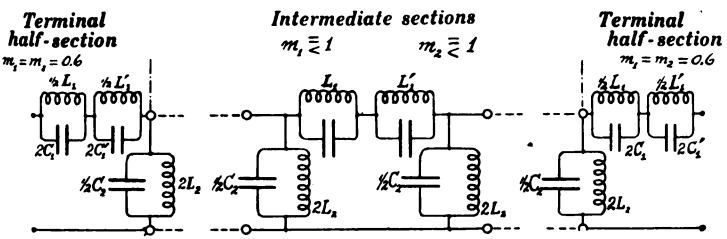
A. DESIGN OF *T*—SECTIONS; BAND-PASS



$$L_1 = m_1 L_{1k} \qquad L_2 = aL_{1k} \qquad L'_2 = cL_{1k}$$

$$C_1 = \frac{C_{1k}}{m_2} \qquad C_2 = \frac{C_{1k}}{b} \qquad C'_2 = \frac{C_{1k}}{d}$$

B. DESIGN OF π–SECTIONS; BAND-PASS



$$L_1 = \frac{L_{2k}}{b} \qquad L'_1 = \frac{L_{2k}}{d} \qquad L_2 = \frac{L_{2k}}{m_2}$$

$$C_1 = aC_{2k} \qquad C'_1 = cC_{2k} \qquad C_2 = m_1 C_{2k}$$

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within 4% of constancy for all frequencies lying in the central 85% of the pass band.

22. Attenuation of lossless composite filters, and the effect of resistance. It was demonstrated in Art. 12 that the attenuation constant α of a composite filter was the sum of the attenuation constants α_1, α_2, etc., of the individual sections. In this summation of attenuation constants, the effect of the two half sections at the ends of the composite filter is the same as the effect of one whole section, for if the central portion of the filter is removed, the two terminal half sections coalesce into one whole section. The attenuation constant of the entire filter is then the attenuation constant of the section made by combining two half sections into one whole section, plus the attenuation constants of the central sections. The attenuation constant of any resistanceless filter section can be computed by equation (iii) of Art. 6.

The ideal resistanceless filter can never exist because every inductance has some resistance, and every condenser has a small though negligible loss. When well-designed condensers and coils make up the filter, however, the losses are so small as to be entirely secondary factors in determining filter characteristics. Some of the effects of losses were considered in Art. 9, where it was stated that losses introduced a slight attenuation of the pass frequencies, tended to round out the abrupt changes of attenuation at the stop frequencies, and required that the load resistance have a negligibly small reactive component to gain perfect action in the pass bands. In addition to these effects, the losses of the composite filter causes the U's of the different sections to differ slightly, thus introducing a very slight reflections of the decaying wave-train at the junctions of the sections. The losses in the filter also make it impossible for any frequency to be attenuated an infinite amount, so that while the attenuation will become very high for frequencies denoted by f_∞ in the design plates, it never reaches infinity.

To actually compute the characteristics of a composite filter

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taking losses into account is a formidable undertaking, and will not be considered here. Fortunately the characteristics of the ideal resistanceless filter are easily determined by equations (iv) and (vi) of Art. 4, by equations (i) and (ii) of Art. 5, and by equations (ii) and (iii) of Art. 6, and the results so obtained are so close to the actual results as to serve for most practical purposes.

23. Summary of method of filter design. The design of a composite filter to meet a given set of conditions falls into several well-defined steps which are briefly covered by the following headings :

First. Determine the cut-off frequencies and the load resistance that will meet requirements, and decide whether T or π-sections are to be used.

Second. Design the terminal half sections as indicated on the proper design plate, using the load resistance and cut-off frequencies selected in step one.

Third. Decide on the number of intermediate sections to be used. The more sections selected, the greater will be the attenuation of stop frequencies. One intermediate section will give a moderately good filter, while two intermediate sections will be sufficient for all ordinary filters. Only in rare cases will the attenuation needed be so great as to require three sections, or more.

Fourth. Select the frequencies of infinite attenuation of the intermediate sections, and then design these intermediate sections for the chosen load resistance, and cut-off frequencies, by the aid of the plate used to design the end half sections. The selection of the frequencies of infinite attenuation must be made with a view of gaining high attenuation for all stop frequencies, and this is done by following the suggestions made in Art. 24.

24. Comments on the design of filters. The first option in the design of filters is a choice between T and π-sections,

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This decision is one of convenience, for electrically there is little or no difference.

The location of the frequencies of infinite attenuation must be carefully selected, for upon this rests the attenuating properties of the filter. These frequencies of high attenuation should be different for the different filter sections, and so staggered throughout the attenuated band that every stop frequency suffers considerable attenuation by at least one section. In this way there is no great surplus of attenuation at some frequencies and no serious lack of attenuation at others. The terminal half sections determine one frequency of infinite attenuation in each stop band, and so should be designed before the intermediate sections in order that the attenuation of these central sections may best supplement the end sections.

Every filter should have a high attenuation for a frequency near each cut-off frequency in order to make the attenuation rise sharply near the boundaries of the pass bands. It is bad practice, however, to make the difference between f_∞ and the cut-off frequency less than about four per cent, for, violating this rule gives a design in which the effect of resistance in the filter inductances is to lower the attenuation of f_∞ from the theoretical infinite value to something very much lower, so low in some cases as to make this a frequency of poor attenuation.

The coils used in the construction of filters must have an inductance that is absolutely definite. For this reason coils with closed iron cores are not allowable because of the varying permeability of iron at different flux densities. The filter inductances must have open iron cores of the kind shown in Fig. 27. If the flux density in the iron does not exceed 30,000 lines per square inch in the iron part of the core, and the air gap offers about ten times as much reluctance as the iron part of magnetic

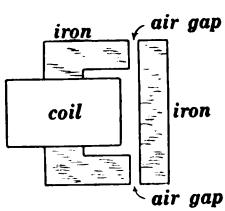

Fig. 27.

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circuit, the coil in Fig. 27 will have low losses, a very nearly constant inductance, and a small size, which are the desired characteristics.

25. Design of a low-pass filter. As an illustration of what can be actually accomplished, the design of a low-pass filter to operate with a load resistance of 500 ohms and to pass

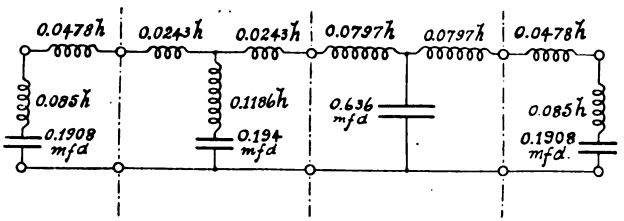

Fig. 28.

frequencies up to 1000 cycles per second will be worked out. The intermediate portion of the filter will consist of two T-sections designed according to Plate IA. The end sections to correspond are determined by the same plate and give $f_\infty = 1250$ cycles. In order to make the attenuation rise rapidly for frequencies near 1000 cycles, one intermediate

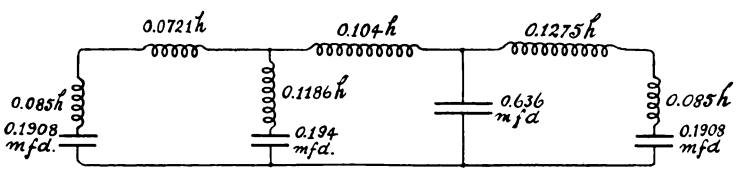

Fig. 29.

section will be designed to give $f_\infty = 1050$ cycles, while the other intermediate section will be designed for $f_\infty = \infty$ in order to give high attenuation at high frequencies. The result is the filter shown in Fig. 28. In this filter it is possible to merge inductances and condensers at the junction points of sections, giving the equivalent filter shown in Fig. 29. The attenuation characteristic of this composite filter is shown in Fig. 30 by the full-line curve, while for purposes of comparison

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the attenuation constant of four similar sections of inverse-network filter are shown by the dotted curve. The superiority of the composite filter in possessing high attenuation near the

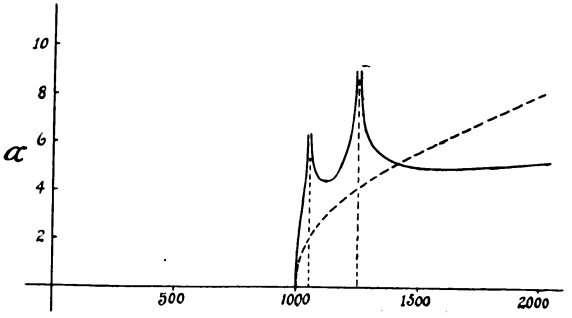

Fig. 30.

cut-off frequency where attenuation is needed, is quite obvious, and the composite filter has the additional advantage of possessing a characteristic impedance U that differs from 500 ohms by less than 4% for frequencies up to 850 cycles.

PROBLEMS

1. Verify the correctness of the filter designed in Art. 24, and shown in Figs. 28 and 29 by computing the required values of inductances and capacities according to the equations in Plate IA.

2. Design the terminal half T-sections of a high-pass filter that has a cut-off frequency of 1000 cycles per second, and is to operate with a load resistance of 500 ohms. Ans. $C_1 = 0{\cdot}2655$ mfd.; $C_2 = 0{\cdot}598$ mfd.; $L_2 = 0{\cdot}0663$ henry.

3. Design an intermediate T-section for the filter in Problem 2. Assume $f_\infty = 900$ cycles per second. Ans. $C_1 = 0{\cdot}366$ mfd.; $C_2 = 0{\cdot}343$ mfd.; $L_2 = 0{\cdot}0914$ henry.

4. Plot the relation between α and frequency for the two-section filter designed in Problems 2 and 3.

5. Design the terminal half π-sections of a high-pass filter that has a cut-off frequency of 1000 cycles per second, and is to operate with a load resistance of 500 ohms. Ans. $C_1 = 0{\cdot}2655$ mfd.; $L_1 = 0{\cdot}149$ henry; $L_2 = 0{\cdot}0663$ henry.

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6. Design an intermediate π-section which has $f_\infty = 900$ cycles per second, and which will be suitable for use with the terminal half sections designed in Problem 5. Ans. $C_1 = 0{\cdot}366$ mfd.; $L_1 = 0{\cdot}0856$ henry; $L_2 = 0{\cdot}0914$ henry.

7. Design the terminal half T-sections of a band-pass filter that is to pass all frequencies between 4000 and 7000 cycles per second, and is to operate with a load resistance of 1000 ohms. Ans. $L_1 = 0{\cdot}0636$ henry; $L_2 = 0{\cdot}00847$ henry; $L'_2 = 0{\cdot}0424$ henry; $C_1 = 0{\cdot}00142$ mfd.; $C_2 = 0{\cdot}0214$ mfd.; $C'_2 = 0{\cdot}1068$ mfd.

8. Design an intermediate T-section which can be operated with the end sections of Problem 7, and for which $f_{1\infty} = 3000$ and $f_{2\infty} = 9000$ cycles per second. Ans. $L_1 = 0{\cdot}0919$ henry; $L_2 = 0{\cdot}0284$ henry; $L'_2 = 0{\cdot}0147$ henry; $C_1 = 0{\cdot}00972$ mfd.; $C_2 = 0{\cdot}0994$ mfd.; $C'_2 = 0{\cdot}02995$ mfd.

9. Apply the principles discussed in Art. 10 to the filters shown in Fig. 12, and show how each of these filters must be of the kind stated in Fig. 12.

10. Explain how the wave filter makes it possible to telegraph over telephone lines by using high-frequency currents, and still prevent the high-frequency telegraph currents and the low voice frequency telephone currents from interfering with one another at the receiving end.

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PART II

HARMONIC ANALYSIS

CHAPTER VI

FOURIER'S SERIES

1. The vibration of a stretched string. It is highly desirable to develope the elementary theory of Fourier's series in connection with a concrete physical problem, and, perhaps, the simplest problem to which Fourier's series can be applied is the problem of the vibration of a stretched string. This problem can, however, be solved in a very simple manner without the use of Fourier's series, and this simple solution is here developed before the Fourier-series solution is taken up.

2. Differential equation of motion of a stretched string. When a stretched string is standing still in equilibrium it is, of course, straight, gravity being neglected. Let us take the equilibrium position of the stretched string as the x-axis of reference as shown in Figs. 1 and 2, and let us set up the differential equation of motion of the string while it is vibrating or while a bend is traveling along it as a wave. We will

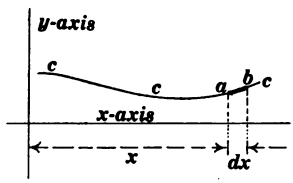

Fig. 1.

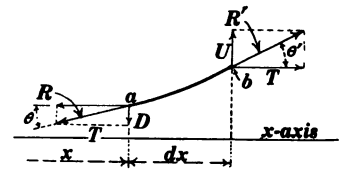

Fig. 2.

consider only the case in which the particles of the string move up and down in Figs. 1 and 2 (motion parallel to the y-axis, no motion whatever parallel to the x-axis), and we will assume that the string is perfectly flexible.*

* This means that the only thing that tends to keep the string straight is its tension.

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Let ccc, Fig. 1, be the configuration of the string at time t, that is to say, ccc is what a photographer would call a snap-shot of the moving string. The shape of the curve ccc defines y as a function of x, and the steepness or slope of the curve at a point is the value of $\frac{dy}{dx}$ at that point.

Consider the very short portion ab of the moving string, the length of this portion when the string lies along the x-axis (in equilibrium) being dx, and the mass of this portion of the string being $m \cdot dx$, where m is the mass per unit length of the string when it is in equilibrium. (It must be remembered that there is assumed to be no motion of the string whatever parallel to the x-axis.) An enlarged view of the portion ab of the moving string is shown in Fig. 2 in which the arrows R and R' represent the forces with which the adjacent parts of the string pull on the portion ab.

Inasmuch as the portion ab of the string is assumed to have no motion parallel to the x-axis and therefore no acceleration parallel to the x-axis, it is evident that the x-component of R must always be equal and opposite to the x-component of R', and this x-component of R (and of R') is, under the assumed conditions, always* equal to the tension T of the string when it stands in equilibrium along the x-axis. Therefore the downward force D which is exerted on the end a of the portion ab is $D = T \tan \theta$, and the upward force U which is exerted on the end b of the portion ab is $U = T \tan \theta'$. Therefore the net upward force which acts on the portion ab of the string is $dF = U - D$ so that

$$dF = T(\tan \theta' - \tan \theta) \quad \ldots\ldots\ldots\ldots \text{(i)}$$

But $\tan \theta$ is equal to the value of $\frac{dy}{dx}$ at a, and $\tan \theta'$ is the value of $\frac{dy}{dx}$ at b; so that the difference, $\tan \theta' - \tan \theta$ is the increase of $\frac{dy}{dx}$ from a to b, and this increase is equal to $\frac{d^2y}{dx^2} \cdot dx$. This is evident when we consider that $\frac{d^2y}{dx^2}$ is the

* Let the student reason this out for himself.

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space-rate of increase of $\frac{dy}{dx}$ (the rate of increase of $\frac{dy}{dx}$ with respect to x). Therefore equation (i) becomes

$$dF = T\frac{d^2y}{dx^2}\cdot dx \quad \text{.....(ii)}$$

Now, according to Newton's second law of motion, the net upward force acting on the portion ab of the string must be equal to the mass $m\cdot dx$ of the portion multiplied by the upward acceleration $\frac{d^2y}{dt^2}$ of the portion. Therefore, substituting $m\frac{d^2y}{dt^2}\cdot dx$ for dF in equation (ii), we get

$$\frac{d^2y}{dt^2} = \frac{T}{m}\frac{d^2y}{dx^2} \quad \text{.....(iii)}$$

which is the required differential equation of motion of the stretched string. It is identical in form to equations (ix) and (x) of Art. 7 of Chapter I, and its general solution is

$$y = f(x - vt) + F(x + vt) \quad \text{.....(iv)}$$

where

$$v = \sqrt{\frac{T}{m}} \quad \text{.....(v)}$$

as explained in Art. 6 of Chapter I.

3. Pure wave traveling towards the right in Fig. 1. The particular solution

$$y = f(x - vt) \quad \text{.....(i)}$$

of equation (iii) of Art. 2 represents what is called a pure wave traveling towards the right in Figs. 1 and 2.

The value of $\frac{dy}{dt}$ at any point of the string at any instant is, of course, the sidewise or transverse velocity V of the string at that point and at that instant. Note that V is upward or downward velocity in Figs. 1 and 2; do not confuse V with the wave velocity v.

The value of $\frac{dy}{dx}$ is, of course, the slope s of the bent string.

Proposition. Any distribution along the string of sidewise velocity V and slope s which satisfies

$$\frac{V}{s} = -v \quad \text{.....(ii)}$$

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constitutes a pure wave traveling towards the right in Figs. 1 and 2.

This proposition is easily established by differentiating equation (i) first with respect to t and then with respect to x which gives

$$V = -vf'(x - vt)$$

and

$$s = f'(x - vt)$$

from which we get (ii); $f'(x - vt)$ represents the derivative of $f(x - vt)$ with respect to $(x - vt)$.

4. Pure wave traveling towards the left in Fig. 1. The particular solution

$$y = F(x + vt) \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

constitutes what is called a pure wave traveling towards the left.

Proposition. Any distribution of sidewise velocity V of the string and of slope s of the string constitutes a pure wave traveling towards the left, if at each point of the string the relation

$$\frac{V}{s} = +v \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

is satisfied.

Remark. Because of the opposite signs in equation (ii) of Art. 3 and equation (ii) of this article *the algebraic sign of v is to be thought of as positive always*; a wave traveling to the right is to be thought of as such, and a wave traveling to the left is to be thought of as such. That is to say, *we are not to think of a wave traveling to the left as a wave traveling to the right at velocity $-v$.* This is indeed desirable from the mathematical point of view, but from the physical point of view it is better to consider the actual facts.

5. Simple examples of a pure wave traveling towards the right in Fig. 1. The simplest distribution of sidewise velocity V and slope s is that in which both are uniform (same in value everywhere), and Fig. 3 represents a wave of this type. The portion AB of the string has uniform slope s (which is a negative slope as shown), and this portion of the string is everywhere moving upwards at velocity V. This wave leaves

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the string permanently displaced sidewise as it travels along the string. Figure 4 shows a wave which does not leave the string permanently displaced sidewise. Let the reader take careful notice of the use of the heavy arrows in Figs. 3 and 4

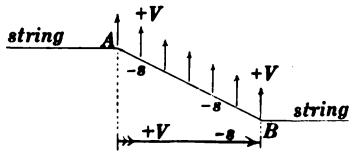

Fig. 3.

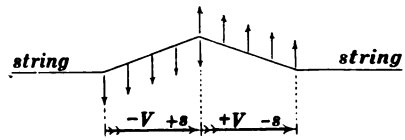

Fig. 4.

for symbolizing the waves so that the following figures may be easily understood.

6. Motion of a long string due to a hammer blow. An infinitely long string is struck by a flat-faced hammer so as to impart a sidewise velocity, $2V$, to the entire portion ab of the string as indicated in Fig. 5. Note that $2V$ is downwards and therefore to be considered as negative. Let the arrow A (Fig. 6) represent a combination of downward velocity $-V$ and of slope $+s$ satisfying equation (ii) of Art. 3, and let the arrow B represent a combination of downward velocity

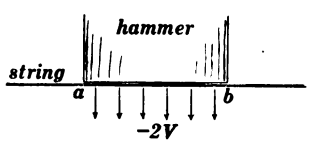

Fig. 5.

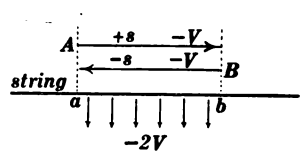

Fig. 6.

$-V$ and slope $-s$ satisfying equation (ii) of Art. 4. Each of these arrows represents a solution of equation (iii) of Art. 2, and both together represent the effect of the hammer blow, namely, zero slope and downward velocity $-2V$ of the portion ab of the string. Now the arrows A and B represent pure waves which travel towards the right and towards the left, respectively, at velocity $v = \sqrt{T/m}$ so that after the lapse of t seconds A will have traveled a distance vt towards

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the right and B will have traveled a distance vt towards the left, and the actual configuration of the string at the instant t (as to sidewise velocity and slope) is easily determined by adding together the sidewise velocities and slopes which are associated with A and B as indicated in Figs. 7 and 8.

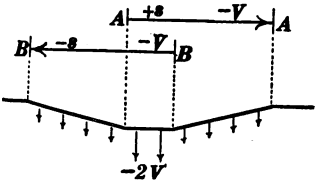

Fig. 7.

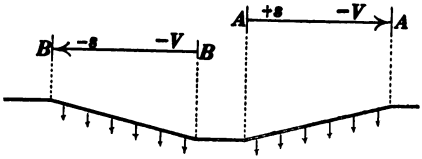

Fig. 8.

7. Satisfying of initial conditions. Any prescribed distribution of sidewise velocity and slope is easily resolvable into two parts such that one part of the sidewise velocity and one part of the slope satisfy equation (ii) of Art. 3 and constitute a pure wave traveling towards the right, and the other part of the sidewise velocity and the other part of the slope satisfy equation (ii) of Art. 4 and constitute a pure wave traveling towards the left; *thereby establishing two particular solutions of equation (iii) of Art. 2 which together satisfy any prescribed initial conditions.* This is exemplified by the above solution of the motion of a string due to a hammer blow.

8. Satisfying of end or boundary conditions. The string in Figs. 5 to 8 is supposed to be infinitely long, and therefore it is not necessary to modify or elaborate the simple solution as given in Art. 6 so as to take care of or express the conditions at the ends of the string. These end conditions can always be taken care of or expressed in terms of the reflection which takes place at the ends, and this matter is here discussed for the two simplest cases, namely, (*a*) when the end of the string is immovable, and (*b*) when the end of the string is entirely free. Case (*a*) corresponds exactly to the *open end* of a transmission line, and case (*b*) which is a highly artificial case for a string corresponds exactly to the *short-circuited end* of a transmission line.

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A pure wave on a string always involves the two elements, namely, *sidewise velocity* and *slope*. Sidewise velocity and slope are *opposite in sign* in a pure wave traveling towards the right, according to equation (ii) of Art. 3; but *alike in sign* in a pure wave traveling towards the left, according to equation (ii) of Art. 4. Therefore when a wave is reflected at either end of a string V or s (but not both V and s) must change in sign because the direction of travel of the wave is reversed by reflection.

Case a. Reflection at immovable end of a string. Let the heavy arrow in Fig. 9 represent* a pure wave approaching

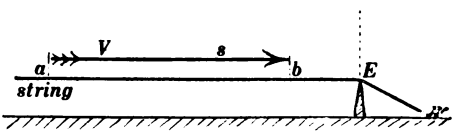

Fig. 9.

the immovable end of the string at E. What happens when this wave reaches E?

Whatever happens *on the string* it must be equivalent to a pure wave traveling towards the right and a pure wave traveling to the left, one heaped on top of the other (superposed), because this is the physical interpretation of equation (iv) of Art. 2 of this chapter, and this equation is the general solution of the differential equation of motion of the string: and whatever happens *at the extreme end of the string* it is only necessary that the actual sidewise velocity of the string be zero there.

Therefore let us assume that a pure wave (sidewise velocity V' and slope s') shoots out from E towards the left when the wave Vs reaches E as indicated in Fig. 10. Now, Vs and $V's'$ being pure waves, we must have

$$\frac{V}{s} = -v \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

and

$$\frac{V'}{s'} = +v \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

* We here think of V and s as having the same values all along ab, but this restriction is not necessary for present purposes.

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according to equation (ii) of Art. 3 and equation (ii) of Art. 4; and because of the immovability of the end E of the string we must have

$$V + V' = 0 \quad \ldots\ldots\ldots\ldots\ldots \text{(iii)}$$

From these three equations we get

$$s' = s \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(iv)}$$

and

$$V' = -V \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(v)}$$

That is to say, *the reflection is complete* because numerically $s' = s$ and $V' = V$; and *the sidewise velocity in the reflected*

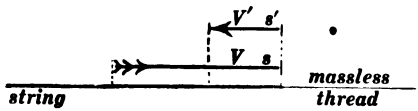

Fig. 10.

Reflection from immovable end of a string.

wave is opposite in sign to the sidewise velocity in the original wave. This is called **reflection with reversal of** V.

Case b. Reflection at the free end of a string. Figure 11 represents an ideal arrangement in which the end of a stretched

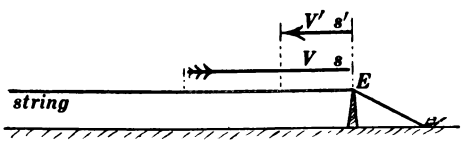

Fig. 11.

Reflection from free end of a string.

string is held by an infinitely long ideal massless thread. Under these conditions the end of the string is perfectly free to move sidewise but the slope of the string at the end must be zero.

Figure 11 shows a given wave Vs being reflected at the free end of the string, and, as in case a, we must have

$$\frac{V}{s} = -v \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(vi)}$$

and

$$\frac{V'}{s'} = +v \quad \ldots\ldots\ldots\ldots\ldots \text{(vii)}$$

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and, as stated above, we must have

$$s + s' = 0 \quad \text{.....(viii)}$$

From these three equations we get

$$V' = V \quad \text{.....(ix)}$$

and

$$s' = -s \quad \text{.....(x)}$$

The reflection is complete and it is called **reflection with reversal of** s.

9. Motion of a plucked string. As an example of a problem involving both initial and boundary conditions let us consider the motion of a string which is plucked at its middle point. The line APB in Figs. 12 and 13 shows the initial

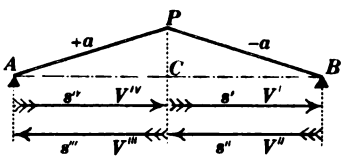

Fig. 12.

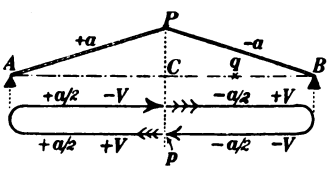

Fig. 13.

shape of the string, slope $+a$ from A to C and slope $-a$ from C to B. This initial condition of the string can be resolved into four pure waves, namely, $V's'$, $V''s''$, $V'''s'''$ and $V^{iv}s^{iv}$ such that

$$s' + s'' = -a \quad \text{.....(i)}$$

$$V' + V'' = 0 \quad \text{.....(ii)}$$

$$s''' + s^{iv} = +a \quad \text{.....(iii)}$$

and

$$V''' + V^{iv} = 0 \quad \text{.....(iv)}$$

and, according to equation (ii) of Art. 3 and equation (ii) of Art. 4, we must have

$$\frac{V'}{s'} = -v \quad \text{.....(v)}$$

$$\frac{V^{iv}}{s^{iv}} = -v \quad \text{.....(vi)}$$

$$\frac{V''}{s''} = +v \quad \text{.....(vii)}$$

$$\frac{V'''}{s'''} = +v \quad \text{.....(viii)}$$

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where $v(=\sqrt{T/m})$ is the known velocity of travel of the waves on the string. Therefore equations (i) to (viii) determine all of the slopes and sidewise velocities, and V and $a/2$ in Fig. 13 represent the common numerical values of all of the sidewise velocities and slopes so determined. We have thus found a set of pure waves which together satisfy the initial conditions.

The two waves $V's'$ and $V''s''$ of Fig. 12 are represented by a single arrow in Fig. 13 because $V's'$ merges into $V''s''$ on reflection (reversal of V); also, and for the same reason, the two waves $V'''s'''$ and $V^{iv}s^{iv}$ of Fig. 12 are represented by a single arrow in Fig. 13. Thus we have resolved the given initial conditions into two pure waves, and the boundary conditions are satisfied by specifying the proper kind of reflection as these two waves travel back and forth along the string. Thus Fig. 14 shows the state of things after each of

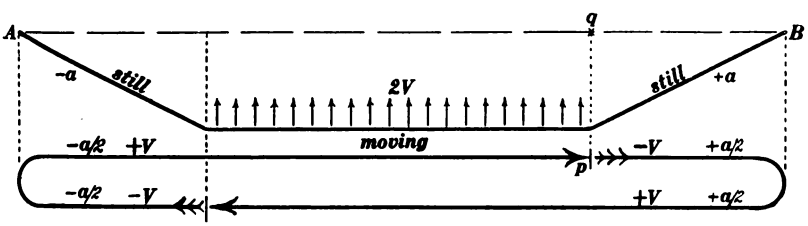

Fig. 14.

the waves in Fig. 13 has traveled over a distance equal to $1\frac{1}{4}l$ where l is the length of the string.

10. Motion of a plucked string as formulated in terms of Fourier's series. A string is pulled to one side as shown in Fig. 15 and released. The string is thus set vibrating, the vibrating string has at each instant a definite shape, that is to say, it forms a definite curve, and it is desired to find an equation expressing y in terms of x and t. This will be at each instant the equation of the curve formed by the string, x and y being co-ordinates as shown in Fig. 15, and t being elapsed time.

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The expression for y in terms of x and t must, of course, satisfy equation (iii) of Art. 2 of this chapter; it must also satisfy the *boundary* or *end conditions*, namely, that $y = o$ at all times when $x = o$ and when $x = \pi$, where π is taken as the length of the string; and it must satisfy the *initial conditions*, namely, the expression for y in terms of x and t must reduce to the equation of the curve formed by the string at the moment of release (when $t = 0$).

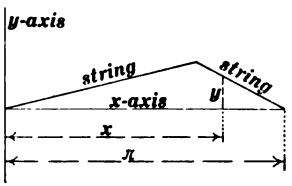

Fig. 15.

Now there is one type of solution of equation (iii) of Art. 2 which satisfies the boundary or end conditions as stated, and this solution is the so-called standing wave-train which is discussed in Art. 5 of Chapter II. The equation of the standing wave-train may be either of the following

$$y = A \sin nx \sin nvt \quad \ldots\ldots\ldots\ldots\ldots \text{(a)}$$

$$y = A \sin nx \cos nvt \quad \ldots\ldots\ldots\ldots\ldots \text{(b)}$$

$$y = A \cos nx \sin nvt \quad \ldots\ldots\ldots\ldots\ldots \text{(c)}$$

or

$$y = A \cos nx \cos nvt \quad \ldots\ldots\ldots\ldots\ldots \text{(d)}$$

Now (*a*) and (*b*), only, give $y = 0$ when $x = 0$; and n must be an integer to give $y = 0$ when $x = \pi$. But (*a*) cannot be used for our present purpose because it gives $y = 0$ everywhere when $t = 0$.

Our problem, therefore, is to take

$$y = A \sin nt \cos nvt$$

as our type-solution, where n may be any positive integer, and superpose a series of such solutions (for $n = 1$, $n = 2$, $n = 3$, etc.) so as to satisfy the prescribed initial conditions. That is to say, we must determine the coefficients A_1, A_2, A_3, etc., so that

$$y = A_1 \sin x + A_2 \sin 2x + A_3 \sin 3x + \ldots,$$

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will be the equation of the curve formed by the string at the moment of release, when $t = 0$. The possibility of doing this was discovered by Fourier* and this possibility is embodied in Fourier's theorem. We will proceed to a discussion of Fourier's theorem and then come back to the problem of the plucked string.

11. Fourier's theorem. Let *ccc*, Fig. 16, be a curve *of any shape whatever* (having but one value of y for each value of x), and let the distance AB be thought of as 2π units [this is equivalent to taking $(AB)/2\pi$ as the unit of length]. Then, according to Fourier, we have

$$\left.\begin{aligned} y = C + A_1 \sin x + A_2 \sin 2x + A_3 \sin 3x + \ldots \\ B_1 \cos x + B_2 \cos 2x + B_3 \cos 3x + \ldots \end{aligned}\right\} \quad \text{..(i)}$$

in which C, A_1, A_2, etc., B_1, B_2, etc., are constants whose values are

$$C = \frac{1}{2\pi}\int_{x=0}^{x=2\pi} y \cdot dx \quad \ldots\ldots\ldots\ldots\text{(ii)}$$

$$A_n = \frac{1}{\pi}\int_{x=0}^{x=2\pi} y \sin nx \cdot dx \ldots\ldots\ldots\text{(iii)}$$

$$B_n = \frac{1}{\pi}\int_{x=0}^{x=2\pi} y \cos nx \cdot dx \ldots\ldots\ldots\text{(iv)}$$

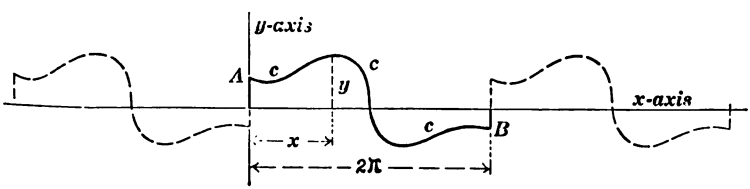

Fig. 16.

Equation (i) is the equation to the given arbitrary curve *ccc* only between A *and* B, and this portion of the arbitrary curve is repeated over and over to the right and left as indicated by the dotted curves in Fig. 16.

A complete proof of Fourier's theorem would involve a

* Fourier's original discussion is very interesting. See Fourier's *Theory of Heat*, translated by Alexander Freeman, Cambridge, 1878.

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proof of the convergence* of the right-hand member of equation (i). This convergence will here be taken for granted, and the proof of the theorem then reduces to the derivation of equations (ii), (iii) and (iv).

In this proof a consideration of average values is most important. For example, let z be a function of x; then the average value of z between x_1 and x_2 is, by definition

$$\left\{\begin{array}{l}\text{average value of } z \\ \text{between } x_1 \text{ and } x_2\end{array}\right\} = \frac{1}{x_2 - x_1}\int_{x_1}^{x_2} z \cdot dx$$

and it is helpful to interpret this definition as follows: The average ordinate of any portion of a curve is equal to the "area under the curve" divided by the base, $x_2 - x_1$. In this statement "area under the curve" means the area between the curve and the x-axis and this area is to be considered as negative when the curve is below the x-axis.

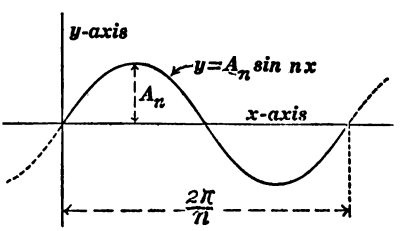

Fig. 17.

Another matter of importance is to understand clearly the meaning of the equation $y = \sin nx$ (and $y = \cos nx$) when n is an integer. Now $y = \sin nx$ is the equation of a sine curve. If $n = 1$ this sine curve makes a positive "arch" between $x = 0$ and $x = \pi$, and a negative "arch" between $x = \pi$ and $x = 2\pi$. In general the sine curve $y = \sin nx$ makes a positive arch between $x = 0$ and $x = \pi/n$, and a negative arch between $x = \pi/n$ and $x = 2\pi/n$ as shown in Fig. 17. Therefore the sine curve $y = \sin nx$ makes n pairs of positive and negative arches between $x = 0$ and $x = 2\pi$.

The derivation of equations (ii), (iii) and (iv) depends upon the following propositions, n and m being integers.

Proposition I. *The average value of* $\sin nx$ (*or of* $\cos nx$) *between* $x = 0$ and $x = 2\pi$ *is zero.* This is evident when we

* This matter is discussed in Chapter III of Byerly's *Fourier's Series and Spherical Harmonics*, Ginn and Co., Boston, 1893.

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consider that $\sin nx$ (or $\cos nx$) passes through exactly similar sets of positive and negative values which exactly offset each other between $x = 0$ and $x = 2\pi$.

Proposition II. *The average value of* $\sin^2 nx$ (*or of* $\cos^2 nx$) *between* $x = 0$ *and* $x = 2\pi$ *is equal to* $\frac{1}{2}$. This may be shown by finding the value of $\frac{1}{2\pi}\int_{x=0}^{x=2\pi} \sin^2 nx \cdot dx$ (or of the corresponding expression for $\cos^2 nx$).

Proposition III. *The average value of* $\sin nx \sin mx$ *between* $x = 0$ *and* $x = 2\pi$ *is zero when* n *and* m *are different integers.* This may be shown by finding the value of the expression

$$\frac{1}{2\pi}\int_{x=0}^{x=2\pi} \sin nx \sin mx \cdot dx ,$$

which may be easily done with the help of a table of integrals.

Proposition IV. *The average value of* $\sin nx \cos mx$ *between* $x = 0$ *and* $x = 2\pi$ *is zero whether the integers* n *and* m *be the same or not.* This may be shown with the help of a table of integrals.

Proposition V. *The average value of* $\cos nx \cos mx$ *from* $x = 0$ *to* $x = 2\pi$ *is zero when* m *and* n *are different integers.* This can be shown with the help of a table of integrals.

Derivation of equation (ii). Consider the average value of each member of equation (i) between $x = 0$ and $x = 2\pi$. The average value of the first member is, by definition, equal to $\frac{1}{2\pi}\int_{x=0}^{x=2\pi} y \cdot dx$, and the average value of the second member is C according to the first proposition above. Therefore we have equation (ii).

Derivation of equation (iii). Multiply both members of equation (i) by $\sin nx$ and consider the average value of the two members of the resulting equation between $x = 0$ and $x = 2\pi$. The average value of the first member is $\frac{1}{2\pi}\int_{x=0}^{x=2\pi} y \sin nx \cdot dx$. The average value of every term of the second member is zero, according to the above propositions,

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except the term $A_n \sin^2 nx$, and the average value of this term is $\frac{1}{2}A_n$ according to the second of the above propositions. Therefore we have equation (iii).

Derivation of equation (iv). Multiply both members of equation (i) by $\cos nx$, consider the average value of both members of the resulting equation between $x = 0$ and $x = 2\pi$, and equation (iv) is obtained.

12. Expression of a portion of any curve as a series of sines. Let *ccc*, Fig. 18, be the given portion of any curve, let the length of this portion be taken as π as shown, and let the second half of the complete wave (wave-length 2π) be chosen like *ccc* turned upside down and turned end for end as shown

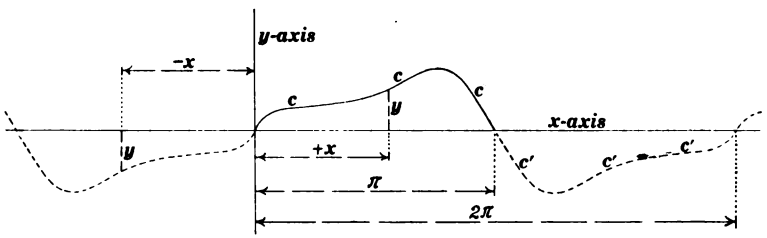

Fig. 18.

by the dotted curve $c'c'c'$. Then the B coefficients in equation (i) of Art. 11 will be zero and equation (i) of Art. 11 will contain sine terms only.

It is to be noted that as expressed by equation (i) of Art. 11 the complete curve *ccc* $c'c'c'$ in Fig. 18 is repeated to right and left, and the kind of symmetry of this periodic curve (for which the B coefficients are zero) is that the ordinates y are the same in value but opposite in sign for $+x$ and $-x$.

13. Expression of a portion of any curve as a series of cosines. Let *ccc*, Fig. 19, be the given portion of any curve, let the length of this portion be taken as π as shown, and let the second half of the complete wave (wave-length 2π) be chosen like *ccc* turned end for end (but not turned upside down) as shown by the dotted curve $c'c'c'$ in Fig. 19. Then the

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A coefficients in equation (i) of Art. 11 will be zero and equation (i) of Art. 11 will contain cosine terms only.

As expressed by equation (i) of Art. 11 the complete curve $ccc\ c'c'c'$ in Fig. 19 repeats itself to right and left, and the kind of symmetry of this periodic curve (for which the A coefficients are zero) is that the ordinates y are the same in value and in sign for $+x$ and $-x$.

14. Conditions under which the constant term C in equation (i) of Art. 11 is zero. This constant term C is equal to the average value of y throughout a complete wave (wave-length 2π) according to equation (ii) of Art. 11. Therefore $C = 0$ when the average value of y throughout a complete

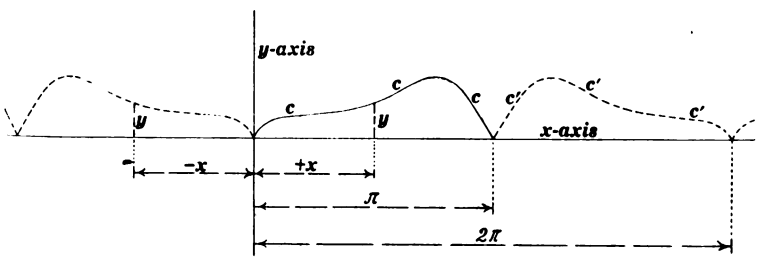

Fig. 19.

wave is zero. Thus the value of C is zero for the curve $ccc\ c'c'c'$ in Fig. 18, but if this whole curve is raised or lowered from the position shown in Fig. 18 the value of C will not be zero.

15. The vibration of a plucked string completely formulated. The problem of the motion of a plucked string was reduced in Art. 10 to the purely mathematical problem of adding together a number of terms like $A \sin nx \cos nvt$ so as to get for y an expression which when $t = 0$ would be the equation to an arbitrary curve, the curve formed by the plucked string at the moment of release. Therefore, in accordance with Art. 12 we will think of the length of the string as π and we will picture a complete wave (wave-length 2π) as indicated in Fig. 20. Then all of the B coefficients are

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zero as explained in Art. 12, and the constant term C is zero as explained in Art. 14 so that equation (i) of Art. 11 becomes

$$y = A_1 \sin x + A_2 \sin 2x + A_3 \sin 3x + \ldots\ldots\ldots \text{(i)}$$

The curve formed by the plucked string at the moment of release ($t = 0$) is made up of three straight lines when the curve is extended over the range 2π as shown in Fig. 20, and

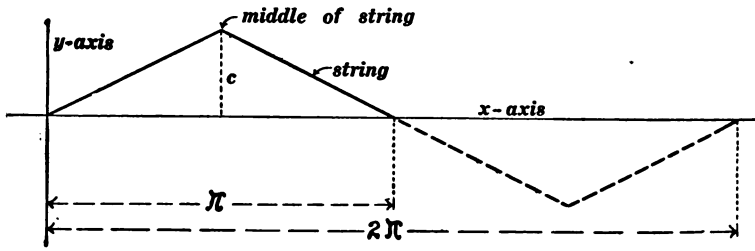

Fig. 20.

if the A-coefficients are to be determined by the use of equation (iii) of Art. 11 the integral must be broken up into three parts; but because of the symmetry of the complete curve as shown in Fig. 20 equation (iii) of Art. 11 is easily reduced to

$$A_n = \frac{4}{\pi}\int_{x=0}^{x=\pi/2} y \sin nx \cdot dx \quad \ldots\ldots \text{(ii)}$$

and between $x = 0$ and $x = \pi/2$ we have $y = (2c/\pi)x$ so that equation (ii) becomes

$$A_n = \frac{8c}{\pi^2}\int_{x=0}^{x=\pi/2} x \sin nx \cdot dx \ldots\ldots \text{(iii)}$$

which gives

$$A_n = \frac{8c}{\pi^2 n^2} \sin \frac{n\pi}{2} \ldots\ldots\ldots\ldots\ldots\ldots \text{(iv)}$$

and if we substitute in equation (i) the values of A_1, A_2, A_3, etc., as determined by this equation we get

$$y = \frac{8c}{\pi^2}(\sin x - \frac{1}{9} \sin 3x + \tfrac{1}{25} \sin 5x - \text{etc}) \ldots \text{(v)}$$

as the equation of the curve formed by the plucked string at $t = 0$, and the equation of the curve formed by the vibrating string at any instant t seconds after the release is

$$y = \frac{8c}{\pi^2} (\sin x \cos vt - \frac{1}{9} \sin 3x \cos 3vt + \text{etc}) \ldots \text{(vi)}$$

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Note. For the kind of symmetry shown by the curve in Fig. 31 (odd-harmonic symmetry) equations (ii), (iii) and (iv) of Art. 11 or equations (ii), (iii) and (iv) of Art. 16 are easily altered so that the integrations may be taken between the limits $x = 0$ to $x = \pi$.

Description of the motion of a plucked string. It would seem from the elaborate difficulties of the above problem [and perhaps the complexity of the result as given in equation (vii)

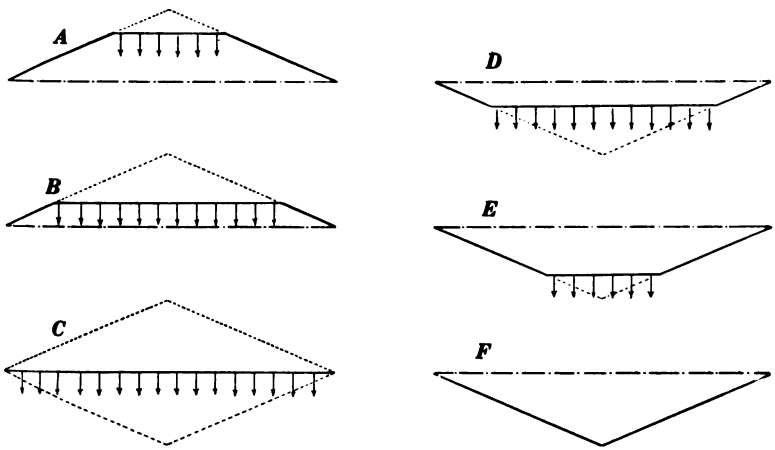

Fig. 21.

would suggest] that the motion of a plucked string is very complicated. As a matter of fact, however, the motion of a plucked string is extremely simple and easy to describe. Thus Fig. 21 shows six successive snapshots of a string which has been plucked in the middle and released. The moving part of the string is straight and parallel to the equilibrium position of the string and the motion is indicated by the short arrows. This character of the motion of a plucked string is easily seen from the discussion given in Art. 9 of this chapter.

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PROBLEMS

1. Determine the coefficients in Fourier's series to give the curve *cc* which is shown in Fig. 22.

Ans. $y = -\frac{2a}{\pi}(\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \ldots)$

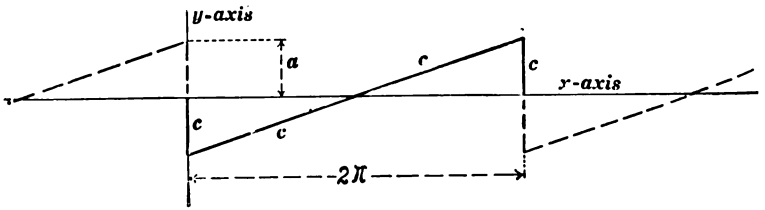

Fig. 22.

2. Determine the coefficients in Fourier's series to give the curve *cc* which is shown in Fig. 23.

Ans. $y = \frac{4a}{\pi}(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots)$

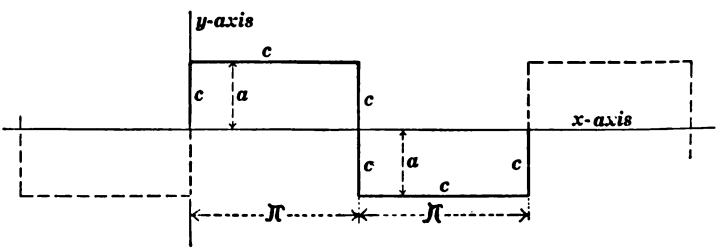

Fig. 23.

3. Determine the coefficients in Fourier's series to give the curve *cc* which is shown in Fig. 24.

Ans. $y = \frac{4a}{\pi}(\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - \frac{1}{7}\cos 7x + \ldots)$

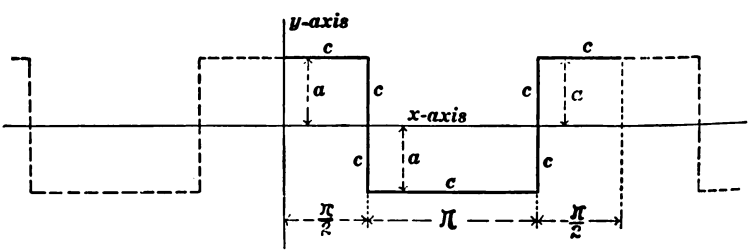

Fig. 24.

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4. Determine the coefficients in Fourier's series to give the curve *cc* which is shown in Fig. 25.

Ans. $y = \frac{8a}{\pi^2}(\sin x - \frac{1}{9}\sin 3x + \frac{1}{25}\sin 5x - \frac{1}{49}\sin 7x + \ldots)$

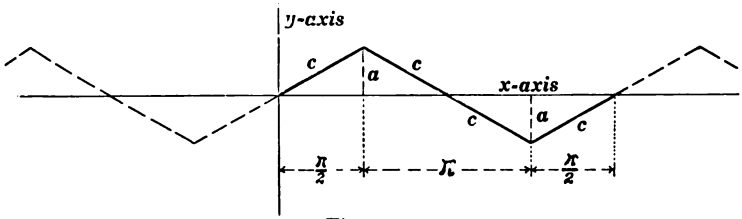

Fig. 25.

5. Determine the coefficients in Fourier's series to give the curve *cc* which is shown in Fig. 26.

Ans. $y = \frac{8a}{\pi^2}(\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \ldots)$

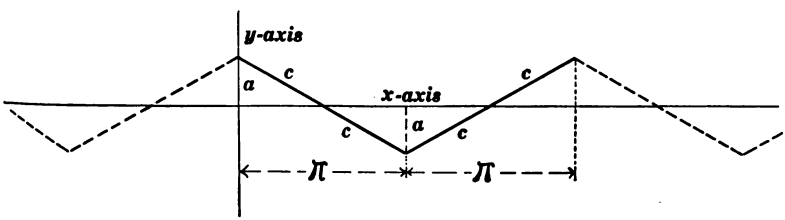

Fig. 26.

6. Determine the coefficients in Fourier's series to give the curve *cc* which is shown in Fig. 27.

Ans. $y = \frac{2a}{\pi}(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \ldots)$

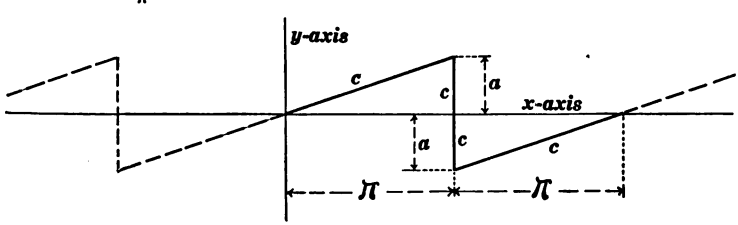

Fig. 27.

7. Derive the answer to problem 4 by integrating the answer to problem 3.

Note. Any Fourier series gives a convergent series when integrated term by term.

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8. Plot the curve which represents the integral of the curve given in Fig. 23, the constant of integration being such as so make the integral curve pass through the origin; and determine the coefficients in a Fourier's series to give the integral curve.

Note. The constant of integration is zero in problem 7 and therefore easily overlooked.

9. Derive the answer to problem 3 from the answer to problem 2 by shifting the origin $\frac{\pi}{2}$ to the right.

10. Differentiate the answer to problem 4 and interpret the result.

Note. A Fourier series for a discontinuous function becomes a divergent series when differentiated, and such a divergent series is meaningless. The curve in Fig. 25 represents y as a continuous function of x. The curves in Figs. 22, 23, 24 and 27 represent y as discontinuous functions of x.

16. A re-statement of Fourier's theorem. Consider any alternating electromotive force or current of which the *period* is T seconds. The ordinates of the periodic curve in Fig. 28 represent such an alternating electromotive force or current, and such an alternating electromotive force or current can be

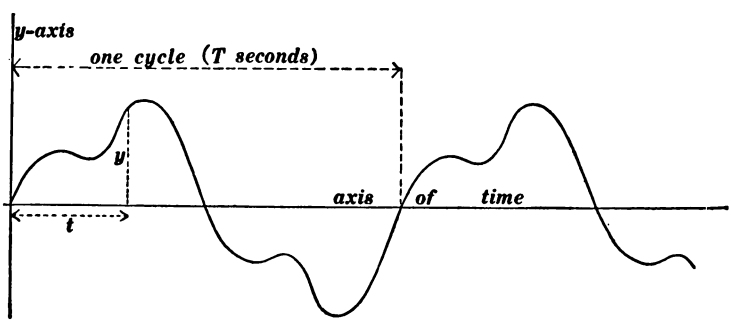

Fig. 28.

resolved into a series of harmonic parts of which the *periods* are T, $T/2$, $T/3$, $T/4$, etc., the maximum value of each part being determinate. Stated in another way, this theorem is as follows: A periodic curve of any shape whatever, for example, the periodic curve shown in Fig. 28, can be exactly matched by superposing a series of sine and cosine curves of

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which the *wave-lengths* are T, $T/2$, $T/3$, $T/4$, etc., where T is the *wave-length* of the given curve.

The harmonic parts into which any given alternating electromotive force or current may be resolved are called the *harmonic components* or simply the *harmonics* of the given electromotive force or current. That harmonic which is of the same frequency as the given alternating electromotive force or current is called the *fundamental harmonic*, and the others are called the *double harmonic*, the *triple harmonic*, the *quadruple harmonic*, etc., in order. It is shown later that only odd harmonics occur in the electromotive force and current curves which are ordinarily met with in practice.

In Art. 11 the wave-length of the periodic curve to be resolved by Fourier's method is taken as 2π units in order to simplify the equations; it is better, however, to write the equations in more general form so as to do away with this limitation. If T is the *wave-length* of a periodic curve (or the *period* of an alternating electromotive force or current) then equations (i) to (iv) of Art. 11 become

$$\left.\begin{aligned} y = C &+ A_1 \sin\frac{2\pi t}{T} + A_2 \sin\frac{4\pi t}{T} + A_3 \sin\frac{6\pi t}{T} + \ldots\ldots \\ &+ B_1 \cos\frac{2\pi t}{T} + B_2 \cos\frac{4\pi t}{T} + B_3 \cos\frac{6\pi t}{T} + \ldots\ldots \end{aligned}\right\} \text{(i)}$$

where
$$C = \frac{1}{T}\int_0^T y \cdot dt \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

$$A_n = \frac{2}{T}\int_0^T y \sin\frac{2\pi nt}{T} \cdot dt \quad \ldots\ldots\ldots\ldots \text{(iii)}$$

and
$$B_n = \frac{2}{T}\int_0^T y \cos\frac{2\pi nt}{T} \cdot dt \quad \ldots\ldots\ldots\ldots \text{(iv)}$$

Remark 1. The average value of an alternating electromotive force or current during a whole cycle is always* zero, and therefore the constant term C in equation (i) is zero when y is the varying value of an alternating electromotive force or current.

* The exceptions to this statement need not be considered here.

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Remark 2. Alternating electromotive force and current curves usually do not have the kind of symmetry possessed by the complete curve *ccc c'c'c'* in Fig. 18 (or Fig. 19), and therefore both sine and cosine terms ordinarily appear in equation (i) when y is the varying value of an alternating electromotive force or current.

Remark 3. See Note in Art. 15.

17. Resultant harmonic of the *n*th order. The two terms of the nth order in equation (i) of Art. 16, namely, $A_n \sin(2\pi nt/T)$ and $B_n \cos(2\pi nt/T)$, are together equivalent to a single harmonic curve because we may write

$$A_n \sin\frac{2\pi nt}{T} + B_n \cos\frac{2\pi nt}{T} = R_n \sin\left(\frac{2\pi nt}{T} + \theta_n\right) \quad \text{(i)}$$

where $$R_n = \sqrt{A_n^2 + B_n^2} \quad \text{(ii)}$$

and $$\tan\theta_n = \frac{B_n}{A_n} \quad \text{(iii)}$$

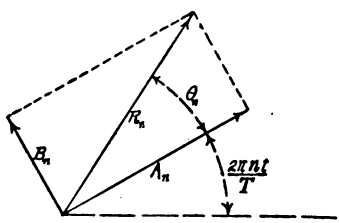

Fig. 29.

This diagram is for the case in which A_n and B_n have the same sign.

This is evident when we consider that $A_n \sin 2\pi nt/T$ and $B_n \cos 2\pi nt/T$ are two harmonic electromotive forces (or currents) in quadrature with each other as shown in the clock-diagram, Fig. 29. In determining the value of θ_n the algebraic signs of A_n and B_n must be duly considered. If it is desired to plot the resultant nth harmonic as shown in Fig. 30, the distance d must be

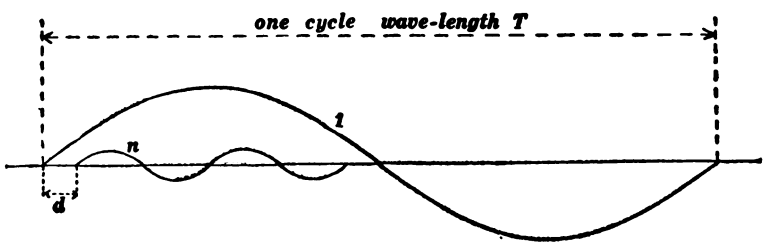

Fig. 30.

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made equal to $\theta_n T/(360n)$, the value of θ_n being expressed in degrees.

18. Even and odd harmonics. Under certain conditions, a periodic curve has odd harmonics only (no harmonic of an

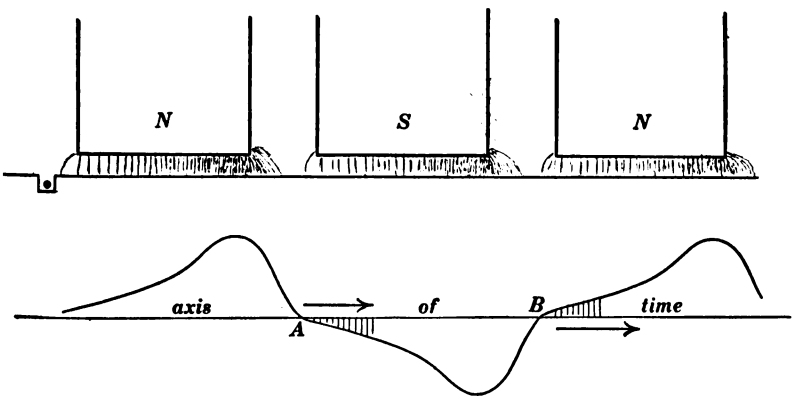

Fig. 31.

order divisible by two), under certain conditions a periodic curve has no harmonic of an order divisible by 3, under certain conditions a periodic curve has no harmonic of an order

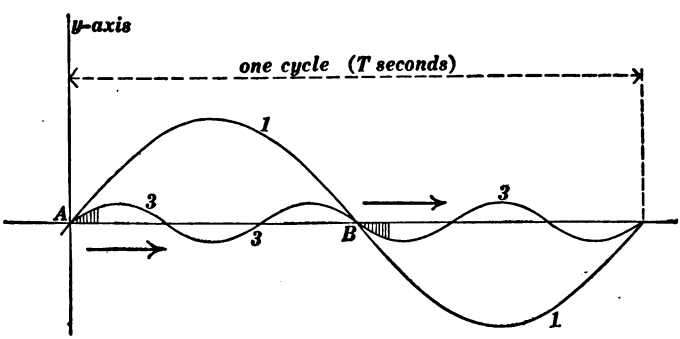

Fig. 32.

divisible by 4, and so on. The most important case for present purposes is the case in which only odd harmonics occur. Thus, *the alternating electromotive force and current curves which*

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occur in practice have only odd harmonics. This is evident from the following considerations :

The north and south poles of an alternator are always alike as shown in Fig. 31. Therefore an alternating electromotive

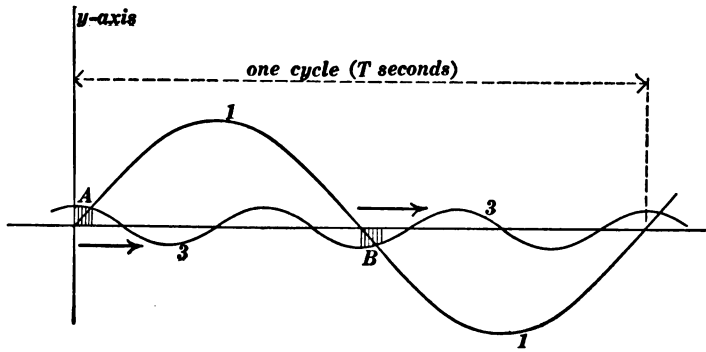

Fig. 33.

force curve always has the kind of symmetry which is shown by the curve in Fig. 31 ; *going towards the right from any two points like A and B half a wave-length apart, equal and opposite values of y occur in order,* as shown by the fine vertical lines

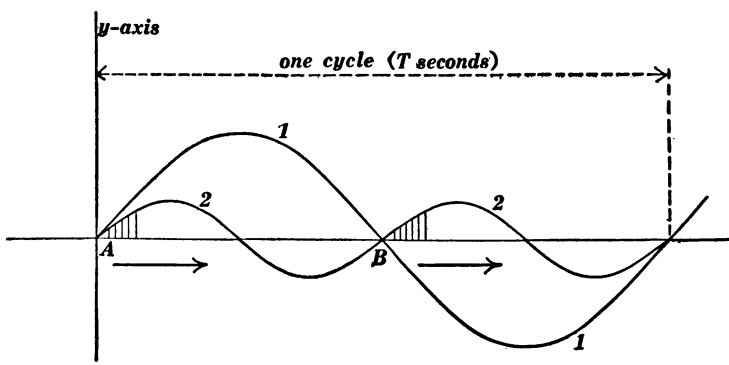

Fig. 34.

in the lower part of Fig. 31. This kind of symmetry, which we will call " odd-harmonic " symmetry, is possessed by all odd harmonics but it is not possessed by any even harmonic. Therefore a curve like that shown in Fig. 31 cannot have an

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even harmonic. Figures 32 and 33 illustrate the "odd-harmonic" symmetry of the sine and cosine components of the triple harmonic (curve 3 in each figure), and Fig. 34 illustrates the kind of symmetry that is possessed by all even harmonics, namely, that equal values of y of the same sign

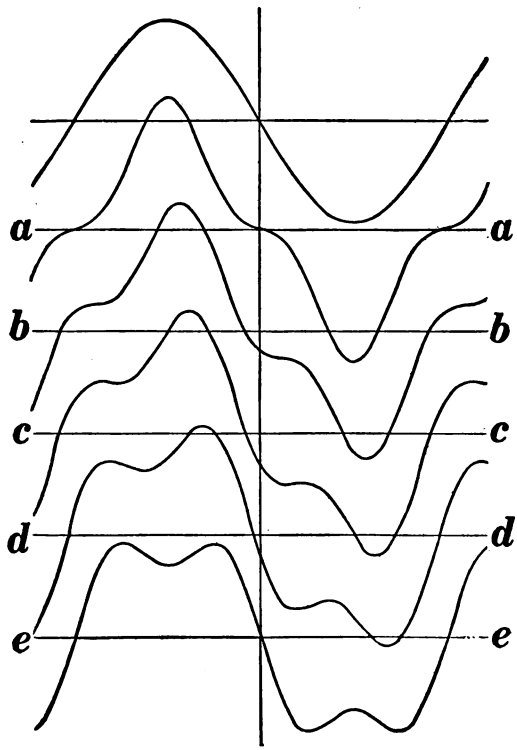

Fig. 35.

occur in order, starting from any two points half a wave-length apart.

Examples. The odd harmonic type of symmetry is possessed by the curves in Figs. 23, 24, 25 and 26 of this chapter and therefore these curves have only odd harmonics.

19. Examples of the distortion of alternating electromotive force curves by triple and quintuple harmonics. The upper curve in Fig. 35 is a simple sine curve, and the curves *aa*,

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bb, *cc*, etc., are obtained when a triple harmonic is added to the given curve. The curve *aa* is obtained when the curve 3 in Fig. 32 is turned upside down, and starting with curve 3 in this position, curves *bb*, *cc*, etc., in Fig. 35 are obtained by shifting curve 3 step by step towards the right in Fig. 32 each step being through a distance $T/12$.*

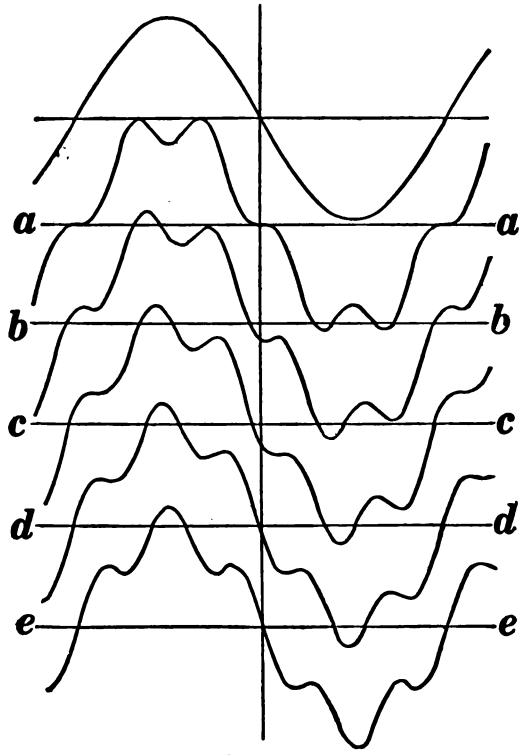

Fig. 36.

The upper curve in Fig. 36 shows a simple sine curve, and the curves *a*, *b*, *c*, *d* and *e* show resultant curves obtained when a quintuple harmonic is added to the given curve. The curve *a* is obtained when the quintuple harmonic passes through zero at the same instant that the fundamental

* The student should plot the fundamental harmonic and the triple harmonic in the various positions to give the resultant curves *a*, *b*, *c*, *d* and *e*.

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harmonic passes through zero, but in a reverse direction, that is, the fundamental changes from negative to positive at the instant that the quintuple harmonic changes from positive to negative; and the successive curves, *b*, *c*, *d* and *e* are obtained by shifting the quintuple harmonic step by step, $\frac{1}{20}$

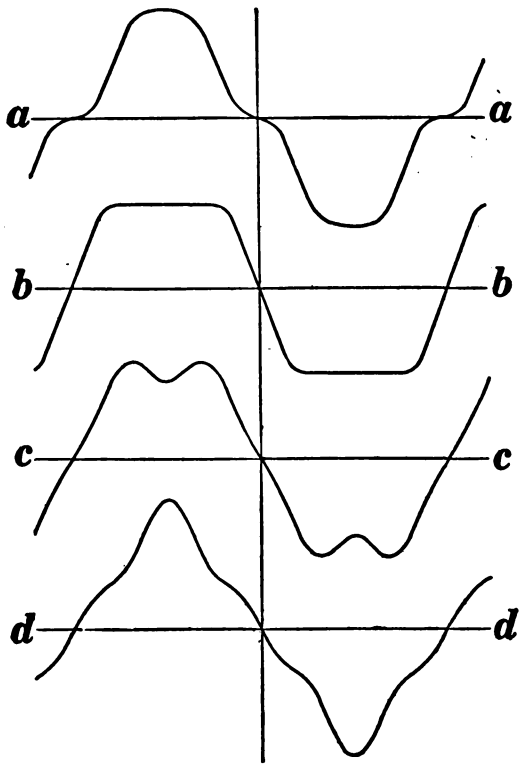

Fig. 37.

of T at each step, where T is the wave-length of the fundamental curve.*

Figure 37 shows some characteristic wave shapes which contain fundamental, triple, and quintuple harmonics. Taking

* The student should plot the fundamental harmonic and the quintuple harmonic in the various positions corresponding to curves *a*, *b*, *c*, *d* and *e*.

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the maximum value of the fundamental harmonic as unity, the equations to these curves are as follows:

Curve *aa*, Fig. 37

$$y = \mathbf{1{\cdot}0} \sin \beta + \mathbf{0{\cdot}15} \sin (3\beta + 180°) + \mathbf{0{\cdot}10} \sin (5\beta + 180°)$$

Curve *bb*, Fig. 37

$$y = \mathbf{1{\cdot}0} \sin \beta + \mathbf{0{\cdot}225} \sin 3\beta + \mathbf{0{\cdot}05} \sin 5\beta$$

Curve *cc*, Fig. 37

$$y = \mathbf{1{\cdot}0} \sin \beta + \mathbf{0{\cdot}15} \sin 3\beta + \mathbf{0{\cdot}10} \sin (5\beta + 180°)$$

Curve *dd*, Fig. 37

$$y = \mathbf{1{\cdot}0} \sin \beta + \mathbf{0{\cdot}15} \sin (3\beta + 180°) + \mathbf{0{\cdot}10} \sin 5\beta$$

20. Effective value of a non-harmonic electromotive force (or current) expressed in terms of its harmonic components. Consider the electromotive force

$$e = A_1 \sin \frac{2\pi t}{T} + A_3 \sin \frac{6\pi t}{T} + \dots \quad \dots\dots\dots \text{(i)}$$

It is desired to find an expression for the square root of the average value of the square (the effective value) of e.

Squaring both members of (i) and indicating averages, we have

$$\left.\begin{aligned} \text{av. } e^2 = A_1^2 \text{ av. } \sin^2\left(\frac{2\pi t}{T}\right) + A_3^2 \text{ av. } \sin^2\left(\frac{6\pi t}{T}\right) + \text{etc.} \\ + 2A_1A_3 \text{ av. } \sin \frac{2\pi t}{T} \sin \frac{6\pi t}{T} + \text{etc.} \end{aligned}\right\} \quad \text{(ii)}$$

Now the average value of $\sin^2 2\pi nt/T$ is one half, and the average value of any cross-product like $\sin 2\pi nt/T \times \sin 2\pi mt/T$ is zero, so that

$$\textit{average } e^2 = \frac{A_1^2}{2} + \frac{A_3^2}{2} + \frac{A_5^2}{2} + \dots \quad \dots\dots \text{(iii)}$$

But $A_1^2/2$, $A_3^2/2$, etc., are the squares of the effective values of the respective harmonics and therefore we have the proposition: *The square of the effective value of an alternating electromotive force or current is equal to the sum of the squares of the effective values of its harmonics.* This proposition is here derived for the special case in which the harmonics are expressible by a series of sines, but the derivation may be easily

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generalised so as to include both the sine series and the cosine series in equation (i) of Art. 16.

21. Proposition. The nth harmonic of voltage and the mth harmonic of current are wattless with respect to each other, except when $n = m$.

This proposition is true in general but we will give its proof only for the simple case in which the fundamentals of voltage and current are in phase with each other and in which sine terms, only, enter.

Consider the electromotive force

$$e = \mathbf{E}_1 \sin \frac{2\pi t}{T} + \mathbf{E}_3 \sin \frac{6\pi t}{T} + \mathbf{E}_5 \sin \frac{10\pi t}{T} + \text{etc.} \quad \ldots\ldots \text{(i)}$$

and the current

$$i = \mathbf{I}_1 \sin \frac{2\pi t}{T} + \mathbf{I}_3 \sin \frac{6\pi t}{T} + \mathbf{I}_5 \sin \frac{10\pi t}{T} + \text{etc.} \quad \ldots\ldots\ldots \text{(ii)}$$

Multiplying (i) and (ii) and indicating averages, we have

$$\left.\begin{aligned} av.\ ei = \mathbf{E}_1\mathbf{I}_1 \text{ av. } \sin^2\left(\frac{2\pi t}{T}\right) + \mathbf{E}_3\mathbf{I}_3 \text{ av. } \sin^2\left(\frac{6\pi t}{T}\right) + \text{etc.} \\ + \mathbf{E}_1\mathbf{I}_3 \text{ av. } \sin \frac{2\pi t}{T} \sin \frac{6\pi t}{T} + \text{etc.} \quad \ldots\ldots\ldots\ldots \end{aligned}\right\} \text{(iii)}$$

but the average value of every cross product like $\sin(2\pi t/T)$ $\sin(6\pi t/T)$ is zero so that the only power terms left in (iii) are those that involve the products $\mathbf{E}_1\mathbf{I}_1$, $\mathbf{E}_3\mathbf{I}_3$, $\mathbf{E}_5\mathbf{I}_5$, etc.

From this fact one derives the important rule that each harmonic contributes its own share of power, as if it were acting alone.

22. Calculation of the harmonics of any experimentally determined curve. The easiest method for determining the coefficients in equation (i) of Art. 11 for any experimentally determined curve is to use the harmonic analyser, one form of which is described in the next article; but it is sometimes necessary to calculate these coefficients when a harmonic analyser is not available. In such cases the so-called schedule method of harmonic analysis is nearly always used.

The schedule method of harmonic analysis is a systematic

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scheme for determining a limited number, N, of the coefficients in equation (i) of Art. 11 so that the curve represented by the equation will pass through N chosen points on the experimental curve.

Let us assume that a given experimental curve has no harmonics of a higher order than the second ($n = 2$). Then the equation of this curve must be

$$y = C + A_1 \sin x + B_1 \cos x + A_2 \sin 2x + B_2 \cos 2x \ldots \text{(i)}$$

and this equation contains five unknown coefficients. Let us choose five points on the given experimental curve whose abscissas are x_1, x_2, x_3, x_4 and x_5; and whose ordinates are y_1, y_2, y_3, y_4 and y_5; respectively. Putting the measured values of the co-ordinates of each point in equation (i) in succession, we get

$$y_1 = C + A_1 \sin x_1 + B_1 \cos x_1 + A_2 \sin 2x_1 + B_2 \cos 2x_1 \quad \text{(ii)}$$
$$y_2 = C + A_1 \sin x_2 + B_1 \cos x_2 + A_2 \sin 2x_2 + B_2 \cos 2x_2 \quad \text{(iii)}$$
$$y_3 = C + A_1 \sin x_3 + B_1 \cos x_3 + A_2 \sin 2x_3 + B_2 \cos 2x_3 \quad \text{(iv)}$$
$$y_4 = C + A_1 \sin x_4 + B_1 \cos x_4 + A_2 \sin 2x_4 + B_2 \cos 2x_4 \quad \text{(v)}$$
$$y_5 = C + A_1 \sin x_5 + B_1 \cos x_5 + A_2 \sin 2x_5 + B_2 \cos 2x_5 \quad \text{(vi)}$$

and these five equations determine the five unknown coefficients.

If the given experimental curve does not, in fact, contain any harmonic higher than the second order, then, using the values of C, A_1, A_2, B_1 and B_2 as determined by equations (ii) to (vi), we will have in equation (i) the correct equation of the experimental curve; but if the experimental curve does contain harmonics higher than the second order, then the curve which is represented by equation (i) will pass through the five chosen points on the experimental curve, but it will depart more or less from the experimental curve between the chosen points. Even so, the values of the coefficients as given by equations (ii) to (vi) will be fairly close approximations to the values as given by equations (ii), (iii) and (iv) of Art. 11 or of Art. 16 if the higher order harmonics are comparatively weak. That is to say, the schedule methods gives fairly

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accurate values of the first N coefficients if the harmonics of higher orders are comparatively weak, especially if the schedule method is carried out on the basis of a large number N of chosen points. For example, using 15 chosen points ($N = 15$) it is possible to determine C, A_1, A_2, ... A_7, B_1, B_2, ... B_7 with a fair degree of accuracy if the higher order harmonics are comparatively weak.

The solution of 15 or more simultaneous equations like equations (ii) to (vi) would ordinarily be very tedious, but if the chosen points are equally spaced, in general, throughout a whole period or wave-length of the experimental curve the labor can be very greatly reduced by using a carefully planned

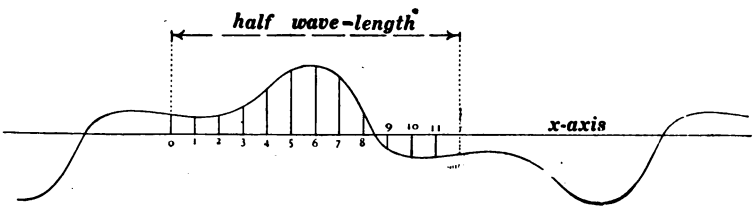

Fig. 38.

schedule. This schedule varies with the degree of symmetry of the experimental curve, but the only case that need be considered here is the case in which $C = 0$ [see equation (i) of Art. 11] and in which only odd harmonics exist.* In this case the schedule method is based on a set of chosen points which divide a half-wave-length of the given curve into an even number N of equal parts. Thus Fig. 38 shows the ordinates that would be chosen for a 12-point ($N = 12$) analysis. It makes no difference where the chosen set of points begins, it is only necessary that they cover a half-wave-length, that is to say, ordinate No. O and ordinate No. N must be half a wave-length apart as shown in Fig. 38.

A twelve-point schedule is shown in the accompanying table. The ordinates are numbered in order. The sums are

* A very full discussion of the Schedule method of harmonic analysis by F. W. Grover on pages 567–646 of Vol. 9 of the *Bulletin of the Bureau of Standards.*

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$s_1 = y_1 + y_{11}$, $s_2 = y_2 + y_{10} \ldots\ldots s_5 = y_5 + y_7$ and $s_6 = y_6$; and the differences are $d_0 = y_0$, $d_1 = y_1 - y_{11}$, $d_2 = y_2 - y_{10}$, etc. The numbers which are tabulated in the columns L, M, N, O, P and Q are not the sums or differences or the quantities σ and δ themselves but the products of these quantities by the corresponding sines in the fourth column. Thus σ_1 in column M is $\sigma_1 \sin 45°$, δ_2 in column P is $\delta_2 \sin 90°$, d_5 in column O is $d_5 \sin 15°$, s_1 in column N is $s_1 \sin 75°$, etc. In the equations which define σ_1, σ_2, δ_1 and δ_2 the letters s and d represent the actual sums and differences as tabulated in the second and third columns.

The six-point and eighteen-point schedules which are given by F. W. Grover on pages 579 and 581 of Vol. 9 of the *Bulletin of the Bureau of Standards* can be followed easily when one is familiar with the twelve-point schedule.

It is important to check the accuracy of the arithmetical work involved in the calculation of the A and B coefficients as explained above, and for the 12-point schedule* the following relations afford convenient checks.

$$y_0 = B_1 + B_3 + B_5 + B_7 + B_9 + B_{11}$$
$$y_6 = A_1 - A_3 + A_5 - A_7 + A_9 - A_{11}$$
$$s_3 = 2(B_1 - B_3 - B_5 + B_7 + B_9 - B_{11}) \sin 45°$$
$$d_3 = 2(A_1 + A_2 - A_5 - A_7 + A_9 + A_{11}) \sin 45°$$
$$d_2 = 2(B_1 - B_5 - B_7 + B_{11}) \sin 60°$$
$$s_4 = 2(A_1 - A_5 + A_7 - A_{11}) \sin 60°$$

23. The harmonic analyser.† Given an experimental periodic curve of which the various harmonics are to be determined, that is to say, the coefficients in equation (i) of Art. 11 or

* Checks for the 6-point schedule and for the 18-point schedule are given in the paper above referred to.

† The Thomson or Kelvin analyser. See Section 37 of the article on *Tides* in the ninth edition of the *Encyclopedia Britannica*. See also articles in the *Proceedings of the Royal Society*, Vol. 24, page 262 and pages 269 and 271.

The Henrici analyser, *Phil. Mag.*, Vol. 38, p. 110, 1894.

The Yule analyser is described by J. N. Le Conte, *Phys. Rev.*, Vol. 7, p. 27, 1898.

The Michelson-Stratton analyser, *Phil. Mag.*, Vol. 45, p. 85, 1898.

The Bush analyser, *Jour. A.I.E.E.*, Vol. 39, p. 309, 1920.

The Dellenbaugh analyser, *Jour. A.I.E.E.*, February, 1921.

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TWELVE-POINT SCHEDULE

Ordinates.		Sums.	Diffs.		For sine coefficients.			For cosine coefficients.		
					A_1 and A_{11}	A_3 and A_9	A_5 and A_7	B_1 and B_{11}	B_3 and B_9	B_5 and B_7
y_0			d_0		(L)	(M)	(N)	(O)	(P)	(Q)
y_1	y_{11}	s_1	d_1	sin 15°	s_1		s_5	d_5		d_1
y_2	y_{10}	s_2	d_2	,, 30°	s_2		s_2	d_4		d_4
y_3	y_9	s_3	d_3	,, 45°	s_3	σ_1	$-s_3$	d_3	δ_1	$-d_3$
y_4	y_8	s_4	d_4	,, 60°	s_4		$-s_4$	d_2		$-d_2$
y_5	y_7	s_5	d_5	,, 75°	s_5		s_1	d_1		d_5
y_6		s_6		,, 90°	s_6	σ_2	s_6	d_0	δ_2	d_0
				Sums ->	(1)	(2) (3)	(4) (5)	(6) (7)	(8) (9)	(10) (11) (12)

$\sigma_1 = s_1 + s_3 - s_5$
$\sigma_2 = s_2 - s_6$
$\delta_1 = d_1 - d_3 - d_5$
$\delta_2 = d_0 - d_4$

$A_1 = \frac{1}{6}[(1) + (2)]$
$A_3 = \frac{1}{6}[(3) + (4)]$
$A_5 = \frac{1}{6}[(5) + (6)]$
$A_7 = \frac{1}{6}[(5) - (6)]$

$A_9 = \frac{1}{6}[(3) - (4)]$
$A_{11} = \frac{1}{6}[(1) - (2)]$
$B_1 = \frac{1}{6}[(7) + (8)]$
$B_3 = \frac{1}{6}[(9) + (10)]$

$B_5 = \frac{1}{6}[(11) + (12)]$
$B_7 = \frac{1}{6}[(11) - (12)]$
$B_9 = \frac{1}{6}[(9) - (10)]$
$B_{11} = \frac{1}{6}[(7) - (8)]$

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Art. 16 are to be determined. Let x be the abscissa and y the ordinate of any point on the given curve and let 2π be the wave-length of the given curve. Then equations (ii), (iii) and (iv) of Art. 11 show that to determine the coefficients we must find the areas under the curves y, $y \cdot \sin nx$, and $y \cdot \cos nx$ between $x = 0$ and $x = 2\pi$, and a harmonic analyser is an integrating machine for finding these areas. The best known harmonic analyser is the one which was suggested by James Thomson and perfected by William Thomson (Lord Kelvin). The integrating part of the machine consists of a circular disk of steel DD, Fig. 39, which is mounted

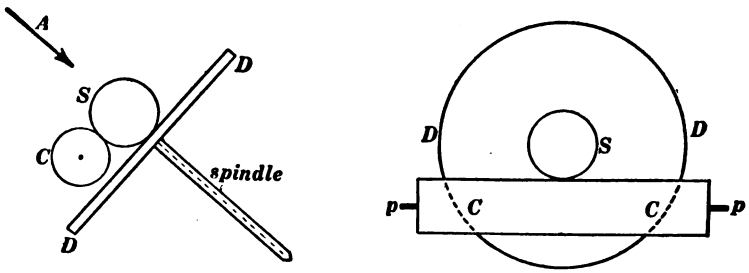

Fig. 39.

upon an inclined spindle as shown, a steel sphere S and a steel cylinder CC which is carried upon two pivots pp. This integrating mechanism is also shown in Fig. 40. The periodic curve which is to be analysed is traced upon paper and wrapped around a cylinder EE, Fig. 40, the circumference of the cylinder being equal to one complete wave-length of the periodic curve. As the cylinder EE is slowly turned, the tracing point P is made to follow one complete cycle of the curve, and the steel fork F pushes the sphere back and forth along the diameter DD of the circular disk so that the distance of the center of the sphere from the axis of the disk is at each instant equal to the ordinate y of the given curve as shown in Figs. 40 and 41.

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This cylinder EE is geared to a series of cranks any one of which may be so connected to the disk DD as to cause it to oscillate back and forth about its supporting spindle as an axis as the cylinder EE is steadily turned. One of the cranks would cause the disk DD to make one complete oscillation while the cylinder EE makes one revolution, the next crank would cause the disk DD to make two oscillations while the cylinder EE makes one revolution, the next crank would cause the disk DD to make three oscillations during one

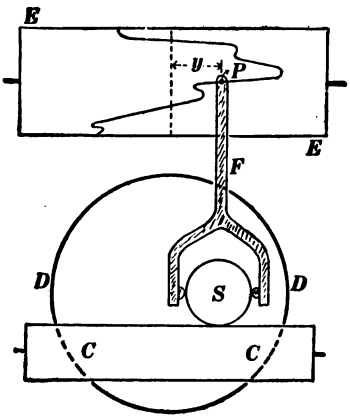

Fig. 40.

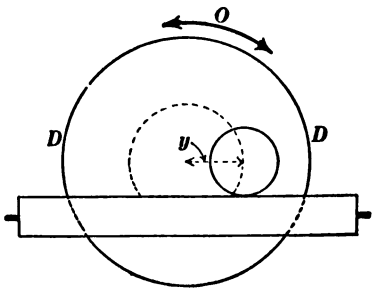

Fig. 41.

revolution of the cylinder EE, and so on. Thus the nth crank would cause the disk DD to make n complete oscillations during one revolution of the cylinder EE. Let T be the time of one revolution of the cylinder EE, then the angular velocity of the oscillating disk at any given instant is equal to $\sin 2\pi nt/T$.* The sphere S rolls on the oscillating disk and against the cylinder CC causing the cylinder to turn at an angular velocity which is at each instant equal to $y/c \times \sin 2\pi nt/T$, where c is the radius of the cylinder CC.

* This means that the disk makes n complete oscillations in T seconds, and it means that the maximum angular distance of the oscillating disk from its mean or middle position is $T/2\pi n$ of a radian.

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Therefore the total angle turned by CC during one complete revolution of the cylinder EE, that is, in T seconds, is

$$\frac{1}{c}\int_0^T y \sin\frac{2\pi nt}{T}\cdot dt$$

Therefore, according to equation (iii) of Art. 16, the coefficient A_n is equal to $2c/T$ times the angle in radians turned by the cylinder CC during one revolution of the cylinder EE, and this angle may be determined by attaching a pointer to the cylinder CC and reading the position of the pointer on a divided circle.

QUESTIONS AND PROBLEMS

11. Find the effective value of an alternating current whose fundamental has a maximum value $\mathbf{I}_1 = 10$ amperes and whose triple harmonic has a maximum value $\mathbf{I}_3 = 2$ amperes; and find the effective value of each harmonic.

12. Carry out the 12-point schedule for the ideal curve shown in Fig. 25 (taking $a = 6$) and compare the values of A_1, A_3, A_5, etc., so found with the correct values of these coefficients as given in answer to problem 4.

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CHAPTER VII

NON-HARMONIC ALTERNATING ELECTROMOTIVE FORCES AND CURRENTS

1. Causes of non-harmonic electromotive forces of alternating-current generators. A full treatment of this subject is of interest chiefly to the designing engineer and it is beyond the scope of this text. The following discussion of a few simple examples is sufficient to set forth the fundamentals.

Figure 1 shows an armature between nearly flat pole faces. If the magnetic flux density at the point b in the gap space

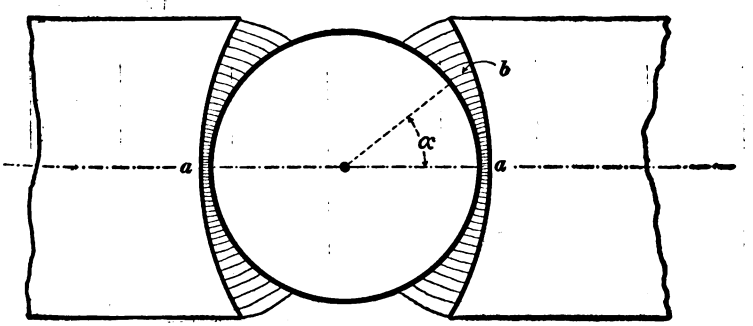

Fig. 1.

were proportional to cos α we would have what is called a harmonically distributed flux. With such a flux distribution a harmonic electromotive force would be induced in the armature windings whether the winding itself be concentrated or distributed.

Figure 2 shows an alternator with uniformly distributed flux in the gap spaces (flux density everywhere the same) and with a fringe of decreasing flux density beyond the pole tips.

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Figure 3 shows the curve of electromotive force induced in a concentrated armature winding by a uniformly distributed flux.

The curves *aa* and *bb* in Fig. 4 are curves of electromotive

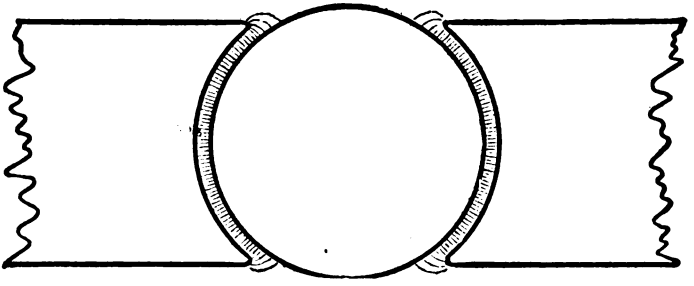

Fig. 2.

force induced in two separate concentrated armature windings *a* and *b*, and the curve *EF* in the lower part of Fig. 4 is the curve of electromotive force induced in the two windings

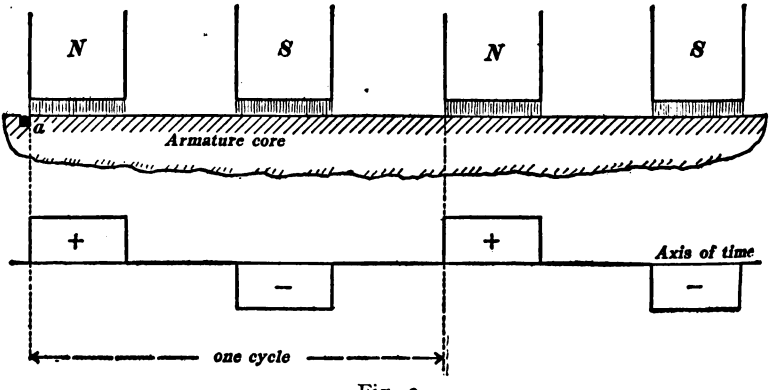

Fig. 3.

a and *b* connected in series, thus giving an armature winding which is distributed in two slots per pole.

The fringes of decreasing flux density beyond the pole tips are ignored in Figs. 3 and 4 so that the drawings may be as simple as possible.

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Figures 5 and 6 show the curves of electromotive force induced in a band of armature inductors, in Fig. 5 the fringes of decreasing flux density beyond the pole tips are ignored; but in Fig. 6 the effect of these fringes is shown, namely, the rounding out of the sharp corners of the electromotive force curve.

Figures 1 to 6 exemplify the influence of the distribution of magnetic flux in the gap spaces and the influence of the

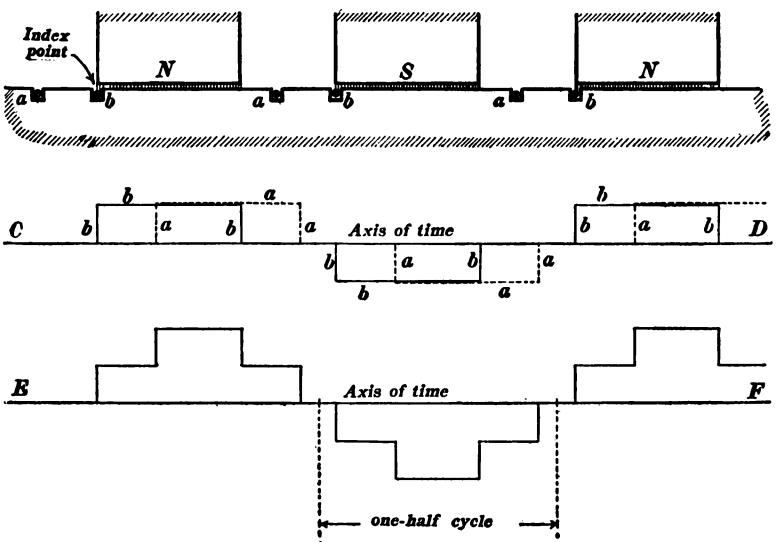

Fig. 4.

distribution of the armature windings on the electromotive force curve of an alternator.

Influence of armature reaction on the electromotive force curve of an alternator. When an alternator is loaded the armature current tends to crowd the field flux under the trailing tips of the pole pieces as shown in Fig. 31 of Chapter VI, and the electromotive force curve of the alternator is distorted as shown by the curve in the lower part of the figure.

The armature reaction of an alternator affects the electromotive force curve of the alternator by distorting the field as

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above explained and also by alternately strengthening and weakening the field as the machine rotates.* As a given armature coil comes into the position in which it surrounds a field pole, the magnetizing action of the current in the coil

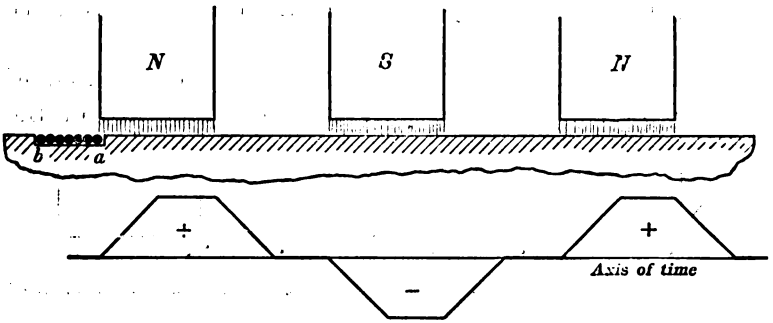

Fig. 5.

either helps or opposes the field flux according as the alternator delivers *leading* current or *lagging* current to its receiving circuit. This magnetizing or demagnetizing action of the armature current is pulsating and it causes the electromotive

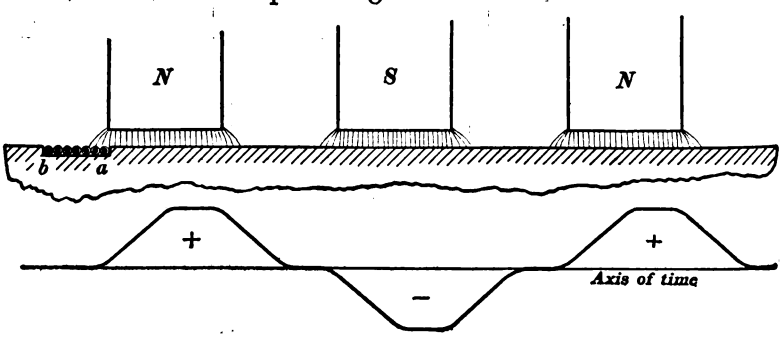

Fig. 6.

force of the alternator to become non-harmonic to some extent.

Pulsation of armature inductance. The inductance of an alternator armature (the inductance of one of its windings if

* This effect is non-existent in a polyphase alternator with a balanced load.

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it is a polyphase armature) varies with the position of the armature. As the armature is turned slowly the inductance reaches a maximum value when the pole pieces bridge over the slots in which the armature winding is placed, and it reaches a minimum value when the slots in which the winding is placed lie between the pole tips. Therefore the inductance of an alternator armature pulsates when the armature rotates. The effect of this pulsating inductance is to cause the electromotive force of the alternator to become to some extent non-harmonic.*

2. Causes of non-harmonic currents. The causes of non-harmonic electromotive forces of alternating-current generators are discussed in Art. 1. The production of a non-harmonic current may depend upon the use of an alternator having a non-harmonic electromotive force, or the reactions of a receiving circuit may cause an alternator having a harmonic electromotive force to deliver a non-harmonic current; and the reaction of a receiving circuit may in turn result in the production of non-harmonic electromotive forces across *a certain portion* of the receiving circuit even though the electromotive force of the generator be harmonic. Therefore, in discussing the production of non-harmonic current, we shall consider first the production of non-harmonic current by a generator of which the electromotive force is non-harmonic, we shall then consider the production of non-harmonic current by a generator having a harmonic electromotive force, and we shall finally consider the production of non-harmonic electromotive forces by the reactions of a receiving system. Before entering upon this discussion, however, it is important to consider constant versus pulsating resistances and inductances.

The current produced in any ordinary non-inductive circuit is proportional to the electromotive force. That is to say, the ratio E/I ($= R$) is a constant for such a circuit. In an electric arc, however, the current is not proportional to the voltage

* Any attempt to formulate the actions described in this article would lead to equations of extreme complexity and the results would not warrant the labor involved.

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across the terminals and therefore the ratio E/I ($= R$) is *not* constant in an arc. An electric arc is said to have a *pulsating resistance.*

The inductance L of a coil may be defined as the quotient Φ/I where Φ is the flux-turns corresponding to a given value of the current I. When the coil does not contain an iron core, Φ is proportional to I, so that the ratio Φ/I ($= L$) is constant. When, however, the coil contains an iron core, Φ is not proportional to I and the ratio Φ/I ($= L$) is *not* constant. In fact, this ratio falls off in value with increasing value of I on account of the magnetic saturation of the iron. Therefore when an alternating current flows through a coil of wire wound on an iron core the value of Φ/I ($= L$) varies as the current rises and falls. A coil with an iron core is therefore said to have a *pulsating inductance.*

The capacity of an ordinary condenser is sensibly a constant and does not pulsate. Therefore when a circuit contains a coil of wire wound on an iron core the reactance $2\pi fL$ pulsates (frequency f constant), whereas the reactance $-1/(2\pi fC)$ of a condenser does not pulsate perceptibly.

3. Non-harmonic currents in circuits of constant resistance and constant inductance. Consider a *non-harmonic electromotive force* of which the values of the fundamental and successive harmonics are A_1, A_2, A_3, A_4, etc. The values of the fundamental and successive harmonics of the current which is produced by this electromotive force through a *non-inductive* circuit are proportional to A_1, A_2, A_3, A_4, etc., and the *current curve is therefore of exactly the same shape as the electromotive force curve.* The values of the fundamental and successive harmonics of the current which is produced by this electromotive force through a highly *inductive circuit* (without iron) are proportional to A_1, $A_2/2$, $A_3/3$, $A_4/4$, etc. The values of the fundamental and successive harmonics of the current which is produced by this electromotive force through a *condenser* (resistance of circuit negligible) are proportional to A_1, $2A_2$, $3A_3$, $4A_4$, etc. The effect of inductance

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is, therefore, to tend to eliminate the higher harmonics of current whereas the effect of a condenser is to exaggerate the higher harmonics of current. This is shown very strikingly

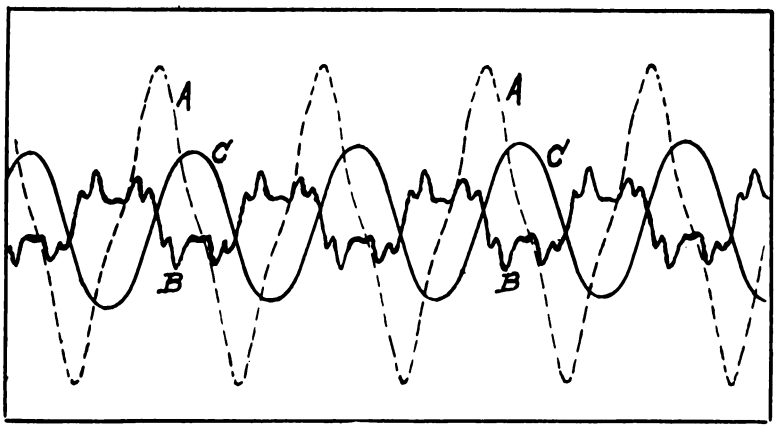

Fig. 7.

in Fig. 7, in which A is a non-harmonic electromotive force curve, C is the curve of the current produced by A in a highly inductive circuit (without iron), and B is the curve of current produced by A through a condenser (resistance of the circuit very small).

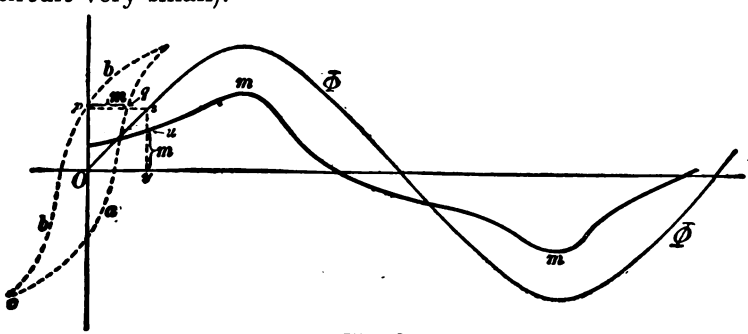

Fig. 8.

4. Production of higher harmonics by reaction of a receiving circuit having pulsating resistance or pulsating inductance. Figure 8 shows the magnetizing-current curve m of a

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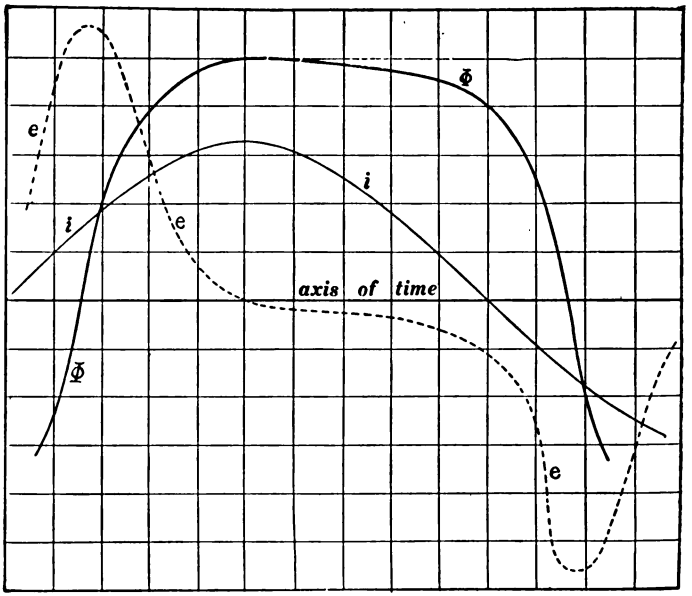

Fig. 9.

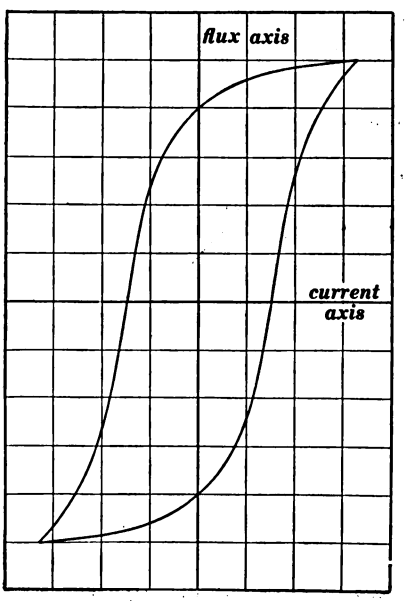

Fig. 10.

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transformer which is supplied from an alternator which gives a harmonic electromotive force. The electromotive force being harmonic, the curve, Φ, Fig. 8, of core flux of the transformer must be harmonic if the resistance of the primary coil is small, and the current corresponding to each value of Φ may be determined from the hysteresis curve of the transformer core (the dotted curve in Fig. 8). That is to say, a *harmonic* electromotive force produces a *non-harmonic* current through a coil of wire on an iron core.

It is possible, however, to produce a *harmonic* current through a winding of wire on an iron core, then the electromotive force across the terminals of the winding is *non-harmonic*. In order to produce a harmonic current through a winding of wire on an iron core (or through an electric arc) it is necessary to use a high-voltage alternator giving a harmonic electromotive force and to include a very large non-inductive resistance in series with the coil (or arc) so that the electromotive force across the coil (or arc) may be negligible in comparison with the total electromotive force of the alternator. Figure 9 shows the electromotive force curve *ee* across a winding of wire on an iron core when a harmonic current flows through the coil. This electromotive force curve may be plotted as follows on the assumption that the resistance of the winding is zero. Having given the hysteresis loop for the given range of current, as shown in Fig. 10, the flux corresponding to each value of the current is known, and the flux curve Φ, Fig. 9, may be plotted. The ordinate of the electromotive force curve is then proportional to the *steepness* of the flux curve at each point of time.

Figure 11 shows the curve of current *i* through an arc lamp and the curve of electromotive force *e* across the terminals of the arc. If the arc lamp were to be connected in series with a very large non-inductive resistance to a very high-voltage alternator giving a harmonic electromotive force, then the current curve would be harmonic as above explained. The curves in Fig. 11 were obtained with a moderate amount

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of resistance and some inductance in series with the arc, and neither the electromotive force curve nor the current curve are harmonic, although the electromotive force of the generator was approximately harmonic.

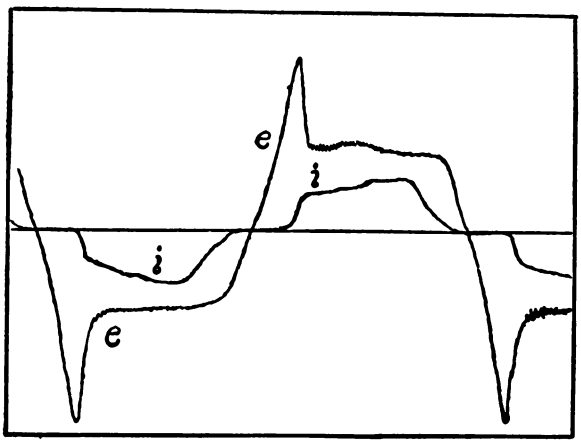

Fig. 11.

5. Phase-order of three-phase electromotive forces and currents. Before discussing the effects of the 3rd, 5th, 7th and higher harmonics in three-phase systems and machines it is necessary to explain what is meant by *phase-order*, and to

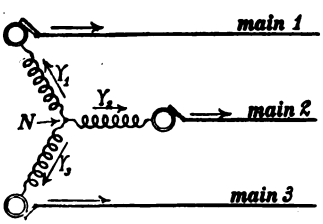

Fig. 12.

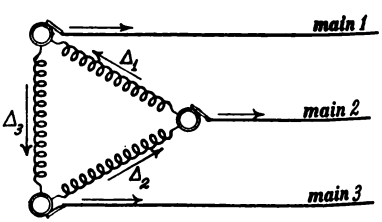

Fig. 13.

examine into the phase relations of the higher harmonics of a system of three-phase voltages or currents.

Definition of phase-order. Figure 12 represents a Y-connected three-phase generator and Fig. 13 represents a Δ-connected three-phase generator; the arrows in each figure

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represent the chosen positive directions in each of the armature windings and in each supply main. The three-phase voltages are represented by Y_1, Y_2 and Y_3 in Fig. 12, and by Δ_1, Δ_2 and Δ_3 in Fig. 13; but in what follows the three-phase voltages will be represented by A, B and C (meaning Y_1, Y_2 and Y_3 in Fig. 12 or Δ_1, Δ_2 and Δ_3 in Fig. 13).

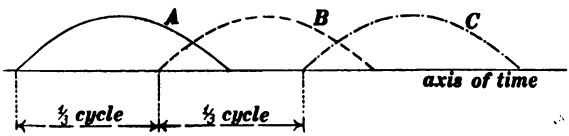

Fig. 14.
Direct phase-order.

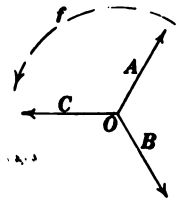

Fig. 15.
Direct phase-order.

Positive half-cycles of the three voltages may occur in either of two ways, namely, as shown in Fig. 14 or as shown in Fig. 16. Figure 14 constitutes what is called the *direct phase-order*, and Fig. 16 constitutes what is called the *inverse phase-order*.

Proposition I. The nth harmonics of three three-phase

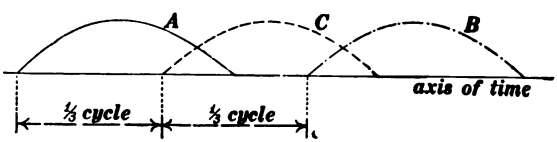

Fig. 16.
Inverse phase-order.

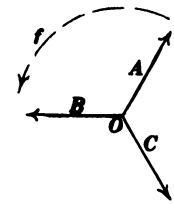

Fig. 17.
Inverse phase-order.

voltages are in phase with each other if n is a multiple of 3. We will discuss this proposition as applied to the voltages Y_1, Y_2 and Y_3 of Fig. 12, and then discuss it as applied to the voltages Δ_1, Δ_2 and Δ_3 of Fig. 13.

Let the three curves A, B and C in Fig. 18 represent the three three-phase voltages Y_1, Y_2 and Y_3, respectively. These curves are supposed *not* to be sine curves but they are

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so shown for the sake of simplicity. These three curves are, however, exactly alike because the three armature windings of a three-phase alternator are alike. Let the three half-waves a_3, b_3 and c_3 be the triple harmonics of A, B and C, respectively. Then, because of the similarity of A, B and C, the distances d are all the same, that is to say, curve a_3 is related to curve A exactly as curve b_3 is related to curve B and exactly as curve c_3 is related to curve C. Therefore, *as shown by the fine-dot curve in Fig.* 18, the three half-waves are parts of one single sine curve, or, in other words, the three harmonics are all positive (directed away from N in Fig. 12)

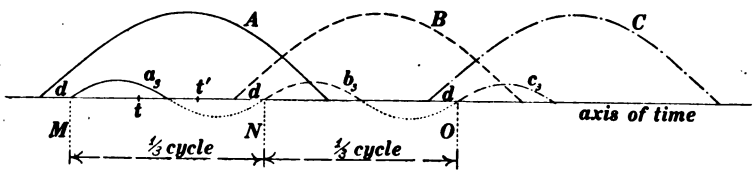

Fig. 18.

together at any instant t, and they are all negative (directed towards N in Fig. 12) together at any instant t'.

Let the three curves A, B and C of Fig. 18 represent the three three-phase voltages Δ_1, Δ_2 and Δ_3 of Fig. 13. Then by an argument exactly like the above it can be shown that the triple harmonics of Δ_1, Δ_2 and Δ_3 are all positive at the same time and all negative at the same time, or, in other words, these triple harmonics *act together* in one direction or the other around the Δ-mesh in Fig. 13, and therefore the three armature windings of a Δ-connected three-phase alternator form a dead short-circuit for the triple harmonics of the three three-phase voltages.

Note. The above discussion is for the case in which $n = 3$, but the same conclusions are reached when n is any multiple of 3. The distances (time intervals) MN and NO in Fig. 18 are equal to $T/3$ where T is the period of each of the given voltages A, B or C. Consider the nth harmonics of A, B

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and C. These harmonics cross the reference axis at equal distances d from the zero-points of the curves A, B and C; their period is T/n; and if T/n is contained a whole number of times in the distance MN it is evident that the curves a_n, b_n and c_n will constitute a single curve of sines, or, in other words, the three nth harmonics will be positive together and negative together (in phase with each other). But if $m \times T/n = MN = T/3$, where m is a whole number, it follows that $n = 3m$, or, verbally expressed, n is a multiple of 3.

Proposition II. If n is divisible by 3 with a remainder of 1,

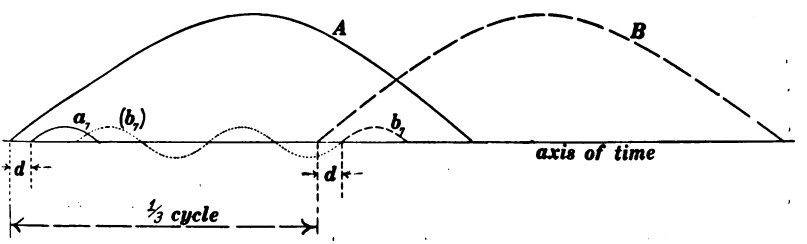

Fig. 19.

then the nth harmonics of given three-phase voltages A, B and C are in the *same phase-order* as A, B and C; whereas if n is divisible by 3 with a remainder of 2, then the phase-order of the nth harmonics of A, B and C is *opposite to* or the *inverse of* the phase-order of A, B and C. Thus the 7th, 13th, 19th, 25th, etc., harmonics of A, B and C are in the same phase-order as A, B and C; whereas, the phase-order of the 5th, 11th, 17th, 23rd, etc., harmonics of A, B and C is opposite to the phase-order of A, B and C.

The general proof of this proposition is not difficult but it is sufficient to give an example instead of the general proof. Thus Fig. 19 shows one half-wave a_7 of the seventh harmonic of A, and one half-wave b_7 of the seventh harmonic of B; and, continuing the curve b_7 as the fine-dot curve, it becomes evident that b_7 is related to a_7 exactly as B is related to A.

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6. The elimination of triple harmonics in a three-wire three-phase system. Some very interesting and important effects are produced in three-wire, three-phase systems because the three-wire scheme of connections makes it impossible for the triple harmonics of the electromotive forces of a Y-connected three-phase generator to contribute to the voltages between mains, and because the armature-windings of a Δ-connected three-phase generator afford a short-circuit for the triple harmonics of the electromotive forces. Also some interesting effects are produced when the receiving circuits of a three-wire, three-phase system produce triple harmonics of electromotive force or current by their reaction, as explained in Art. 4.

Y-connected generator. The discussion in Art. 5 shows that the triple harmonics of Y_1, Y_2 and Y_3 in Fig. 12 are in phase with each other and therefore these triple harmonics annul each other between the supply mains in Fig. 12. *The nth harmonics of the three voltages of a Y-connected three-phase alternator do not show themselves in the voltages which are transmitted by a three-wire line, when n is a multiple of* 3.

Δ-connected generator. The discussion in Art. 5 shows that the triple harmonics of Δ_1, Δ_2 and Δ_3 in Fig. 13 are dead, short-circuited by the mesh-connections of the three armature windings and therefore these triple harmonics do not show themselves in the voltages between the supply mains in Fig. 13. *The nth harmonics of the three voltages of a Δ-connected three-phase alternator do not show themselves in the voltages which are transmitted by the three-wire line, when n is a multiple of* 3.

7. Appearance of triple* harmonics in balanced three-phase receiving systems. It is pointed out in Art. 6 that triple harmonics in a three-phase generator are not transmitted by a three-wire line. Such triple harmonics cannot therefore influence the receiving circuits to which the three-wire line delivers current. When the receiving circuits contain pulsating

* Any set of nth harmonics where n is a multiple of 3.

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resistances or pulsating reactances, however, higher harmonics of voltage and current are produced thereby, including triple harmonics, and the triple harmonics so produced have no influence on the three-wire line or on the generating machinery. In the following discussion the line voltages and line currents are assumed to be simply harmonic, that is, to consist of fundamental sine waves of electromotive force and current without any higher harmonics, and the discussion is limited to the effects of pulsating reactances (primary coils of trans-

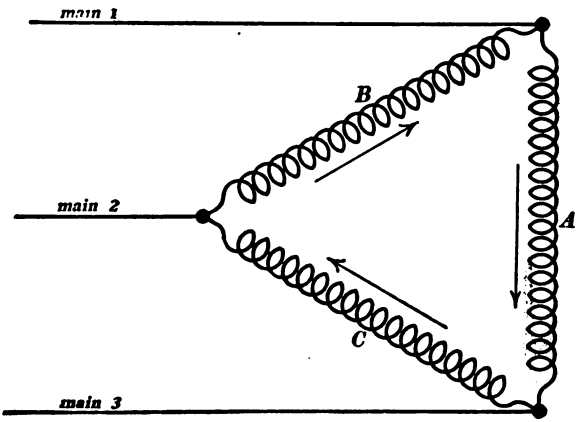

Fig. 20.

formers). A very slight modification would adapt the discussion to the effects of pulsating resistances.

Delta-connected receivers. Let A, B and C, Fig. 20, be three primary coils of three transformers which are supplied over a three-wire, three-phase line, and let us consider the magnetizing currents of the three transformers. The voltage across each transformer being harmonic, the magnetizing current must be non-harmonic as explained in Art. 4. Consider the triple harmonics of the three magnetizing currents. These triple harmonics are all in the directions of the arrows in Fig. 20 or all in the opposite directions simultaneously. Therefore the triple harmonics of the magnetizing currents of three

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Δ-connected transformers circulate back and forth around the short-circuit formed by the Δ-connections.

Y-connected receivers. Let A, B and C, Fig. 21, be the three primary coils of three transformers supplied over a three-wire, three-phase line. The magnetizing currents of the three transformers must be simply harmonic* because triple harmonics of current would have to flow all in the directions of the arrows or all in the reverse direction simultaneously which is obviously impossible. Therefore the electromotive

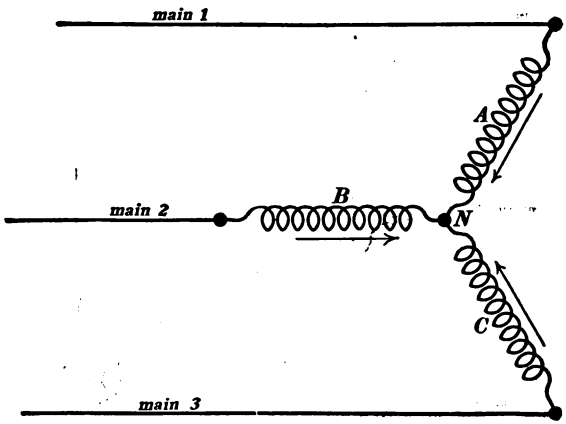

Fig. 21.

forces across the transformer primaries must be non-harmonic as explained in Art. 4. This non-harmonic character of the electromotive forces across the primaries of Y-connected transformers consists chiefly of the presence of triple harmonics which of course balance out between the line wires so that no effect of these harmonics is transmitted back to the generator.

8. The 4-wire, three-phase line. Consider a Y-connected three-phase generator delivering current to Y-connected receiving circuits over a three-wire line as shown in Fig. 22 (fourth main as represented by the dotted line supposed to be,

* Harmonics of 5th and higher orders (not divisible by 3) are not here considered.

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for the moment, non-existent). The generator voltages may contain triple harmonics without influencing the line voltages or the line currents, and triple harmonics may be created by pulsating reactance in the receiving circuits without affecting the line voltages or the line currents, as already explained.

If a 4th main is connected, as indicated by the dotted line in Fig. 22, then the phases become independent of each other,* each phase being transmitted over its own individual circuit.

Suppose, for example, that the generator voltages A_0, B_0 and C_0 contain no triple harmonics, whereas triple harmonics of electromotive force are introduced into the voltages A_1,

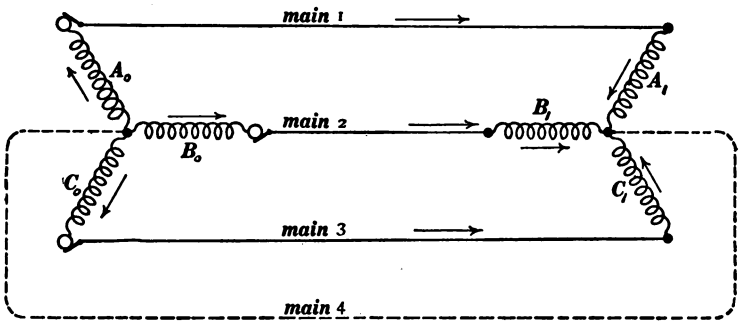

Fig. 22.

B_1 and C_1 as explained in Art. 7. Then the triple harmonics of A_1, B_1 and C_1 will produce triple frequency currents over the three mains 1, 2 and 3 and over main 4; or, if both neutral junctions (of generator and of receiver) are grounded, the triple frequency currents over 1, 2 and 3 will have a ground return.

If one neutral junction, only, is grounded, say the neutral junction of the generator, then a voltmeter connected from the other neutral junction to ground will show the value of the triple harmonic of $A_0 + A_1$ (which is of course the same as the triple harmonic of $B_0 + B_1$ or of $C_0 + C_1$); this voltage between a "neutral" junction and ground may amount

* If the resistance and reactance of the 4th main are negligible.

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to several thousands of volts in the case of a high voltage transmission system. Now, according to the simple working theory of alternating currents (electromotive forces and currents assumed to be harmonic, or, in other words, effects of higher harmonics ignored), there would be no electromotive force between the neutral junction of Y-connected receiving transformers and ground when the neutral junction of the step-up transformers at the generating station is grounded, and an inexperienced operating engineer who is guided wholly by the simple working theory is likely to be ignorant of the danger which is involved in the existence of this triple-frequency voltage between a three-phase "neutral" and ground.

9. The influence of triple harmonics on the current and voltage relations in three-wire, three-phase systems. On the assumption that voltages and currents are all simply harmonic, that is, consisting of the fundamental sine waves alone without higher harmonics, it is shown in elementary treatises that the voltage across the terminals of a Y-connected three-phase generator is equal to $\sqrt{3}$ times the voltage of one armature winding, and that the current in one line is $\sqrt{3}$ times the current in one armature winding of a Δ-connected three-phase generator.

When a three-phase alternator is Y-connected to a three-wire line the presence of triple harmonics in the electromotive forces induced in the armature windings tends to make the voltages between the mains less than $\sqrt{3}$ times the voltage across one armature winding because the triple harmonics which are present in the armature windings are balanced out between the mains.

When a three-phase alternator is Δ-connected to a three-wire line the presence of triple harmonics of current in the windings tends to make the current in each armature winding more than $1/\sqrt{3}$ of the current in each main because the triple harmonics of current circulate only in the Δ-mesh formed by the armature windings.

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10. Influence of triple harmonics upon the power factor of a three-phase generator which supplies currents over a three-wire line. A three-wire distributing system constitutes an open circuit with reference to the triple harmonics of electromotive force of a Y-connected three-phase generator; and therefore the power delivered by each winding of such a generator to a non-inductive receiving system is not equal to EI where E is the electromotive force across the winding, as measured by a voltmeter, and I is the current in the winding as measured by an ammeter. The actual power delivered by the winding is less than EI. That is to say, the power factor of the generator is less than unity.* That is, the output capacity of a Y-connected three-phase generator may be slightly increased by providing a fourth line wire as shown in Fig. 22, if triple harmonics exist.

When a three-phase generator having triple harmonics of electromotive force is Δ-connected to a three-wire line its output capacity is less than it would be if four line wires were used as in Fig. 22 for two reasons, namely, first the short-circuit triple harmonic of current reduces the available voltage of each winding by annulling the triple harmonic of electromotive force by armature reaction, and second the short-circuit triple harmonic of current adds to the heat generated in the armature windings, and the capacity rating of the machine is thereby reduced.

11. Effects of higher harmonics on the behavior of alternating-current machines. Nearly the whole of alternating-current engineering is based upon a simple working theory in which the alternating electromotive forces and currents are assumed to be harmonic; as a matter of fact, however, the electromotive forces and currents are never exactly harmonic, and therefore the behavior of the various alternating-current

* The student should be able to derive an expression for the power factor in terms of the effective values of the fundamental and triple harmonics of the generator electromotive forces, higher harmonics being ignored. See Arts. 20 and 21 of Chapter VI.

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machines departs to some extent from the simple working theory. This departure or difference in behavior due to the non-harmonic character of electromotive forces and currents may be spoken of as the effect of higher harmonics. It is not considered in practical work except in a few instances.

In general, the details of behavior of an alternating-current machine of any kind when electromotive forces and currents are non-harmonic may be thought of as the combination or summation of the different effects which would be produced by the several harmonics taken singly. This is strictly true in those cases in which an exact proportional relationship exists between electromotive force and current (when resistances and inductances do not pulsate). When, however, the system contains arc lamps or inductances with iron cores, then the effects of the various harmonics taken singly cannot be added together to give the effects of all together. For practical purposes, however, the effects of non-harmonic electromotive forces and currents may usually be thought of as the summation of the effects which would be produced by the various harmonic components taken singly.

In the following discussion of the synchronous motor and induction motor it must be remembered that all harmonics of three-phase supply voltages are eliminated whose orders are divisible by three, because a three-wire line does not transmit such harmonics.

12. The synchronous motor. Consider an alternating-current generator delivering current to a similar machine acting as a synchronous motor. The action of the two machines due to the fundamental or to any particular harmonic of their electromotive forces is that which is described in the ordinary discussion of the theory of the synchronous motor.

Suppose, however, that the generator has a harmonic which does not normally exist in the motor, the quintuple harmonic, for example. Then this quintuple harmonic of the generator is not opposed by a corresponding quintuple harmonic in the motor, so that the armature of the motor constitutes a short-

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circuit in so far as this quintuple harmonic is concerned. This quintuple voltage harmonic (of the generator) produces therefore a short-circuit quintuple harmonic of current; but it must be remembered that when an alternator is short-circuited, the short-circuit current is limited mainly by the very large inductance of the alternator armature, and the power which is represented by the short-circuit current is not excessively large. In the case of the quintuple harmonic the short-circuit current is even more effectively limited by the inductance of the two alternator armatures because of the higher frequency, so that the short-circuit quintuple-harmonic current is not very large, and it represents but a very small additional power output of the alternator because it is nearly 90° behind the quintuple harmonic of electromotive force in phase.

Another way of looking at the effect of a non-harmonic electromotive force of an alternating-current generator which drives a synchronous motor is as follows: Suppose that an ordinary alternator is short-circuited through a zero resistance. The short-circuit current in the alternator armature may be thought of as modifying the field flux in such a way as to reduce the actual induced electromotive force in the alternator armature to that value which is capable of producing the actual short-circuit current through the armature. In the same way, the short-circuit quintuple harmonic current which is described in the foregoing paragraph may be thought of as modifying the flux distribution of the generator so as to cause a great reduction in the value of the quintuple harmonic in the generator electromotive force curve, and as modifying the flux distribution of the synchronous motor so as to introduce an opposing quintuple harmonic in its electromotive force curve. If the resistance of the circuit were zero, then the quintuple harmonic electromotive force which is introduced into the electromotive force curve of the motor would exactly balance the reduced quintuple harmonic in the electromotive force curve of the generator, and the short-circuit quintuple-harmonic current would be the amount of quintuple-harmonic

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current required to so modify the field fluxes of the two machines as to bring about this result.

13. The induction motor. The behavior of the induction motor due to the fundamental harmonic of the supply voltage is the behavior which is described in the ordinary discussion of the elementary theory of the induction motor. The following discussion applies to the three-phase induction motor.

Consider the 7th harmonics of the supply voltages. These have the same phase-order as the fundamental harmonics, and they produce their own very high-speed rotating stator magnetism. With respect to this very high-speed stator magnetism the rotor runs at a very great slip so that the torque contributed by the 7th harmonics is very small.

Consider the 5th harmonics of the supply voltages. These have a phase-order which is the reverse of the phase-order of the fundamental harmonics, and they produce their own *backward rotating* stator magnetism which produces a very slight backward torque on the rotor.

The 5th and 7th sets of harmonics both cause a slight and somewhat wasteful RI^2 loss of power, because with respect to both these sets of harmonics the slip is very great.

14. Transformer core losses. The maximum flux density reached in the core of a transformer is proportional to the area under the half-wave of the electromotive force curve, the smaller this area the lower the value of the maximum flux density and consequently the lower the hysteresis loss in the iron core for a given frequency. Consider two kinds of electromotive force curve, a flat-topped curve and a peaked curve, the effective values being the same, say, 1000 volts. The flat-topped electromotive force curve will have a greater area under its half-wave than the peaked electromotive force curve. Therefore, the hysteresis loss in a transformer core is less when the alternating electromotive force curve is peaked than it is when the alternating electromotive force curve is flat, both electromotive forces having the same effective value and the

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same frequency. The eddy current loss, on the other hand, depends only upon the effective value of the electromotive force.

15. Production of violent line oscillations by higher harmonics. When the proper frequency of oscillation of a transmission line coincides, or very nearly coincides, with the frequency of the generator electromotive force, a violent state of line oscillation is built up by resonance as explained in Art. 6 of Chapter II. When the distant end of the line is open the length of the line must be an odd number of quarter-wave-lengths to give resonance, and when the distant end of the line is short-circuted the length of the line must be an even number of quarter-wave lengths to give resonance, as explained in Art. 6 of Chapter II. Therefore if a frequency of 25 cycles per second is used the quarter-wave-length (on an air line) would be 1860 miles and this would be the shortest air line for which excessive line oscillations would be set up. It is evident, therefore, that no existing transmission line (the longest of which is only two or three hundred miles) will be set into violent oscillation by an electromotive force whose frequency is 25 cycles per second; but the conditions of resonance may be satisfied by an existing line with respect to one of the higher harmonics of the 25-cycle generator. See Problem 9.

QUESTIONS AND PROBLEMS

1. The armature inductors a and b in Fig. 4 are supposed to be moving towards the right and the inductors are at the positions shown in the figure at a certain instant which we will take as the origin of time ($t = 0$). Furthermore, we will choose inductor b which is farthest towards the left as the moving point (index point) whose position represents elapsed time t. Then elapsed times along CD are the projected positions on CD of the chosen "index point" as it moves towards the right. Where is the instant $t = 0$ on CD? Where is the point on CD which represents the instant

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t when the inductor a (farthest towards the left) reaches the tip of the N-pole ?

2. An alternating electromotive force contains a fundamental whose maximum value is 100 volts, a triple harmonic whose maximum value is 20 volts, and a quintuple harmonic whose maximum value is 10 volts. What are the maximum values of fundamental, triple and quintuple harmonics of current produced through: (*a*) A pure inductive reactance (resistanceless) whose reactance value corresponding to the fundamental is 4 ohms, (*b*) This inductive reactance connected in series with a resistance of 3 ohms, and (*c*) The specified inductive reactance and resistance connected in series with a condenser whose reactance value corresponding to the fundamental is 10 ohms ?

3. Why must a harmonic electromotive force when connected to the primary coil of a transformer produce a harmonically varying core flux if the resistance of the primary coil is negligible ?

4. Draw with some care an arbitrary hysteresis loop showing the relation between the cyclic magnetizing current and the values of the core flux of a transformer, and construct from this hysteresis loop the curve of magnetizing current (abscissas representing elapsed time) to give a harmonically varying core flux.

5. Draw with some care an arbitrary hysteresis loop (see Fig. 10) and construct the curve of the electromotive force which must be applied to a primary transformer coil of zero resistance to give a harmonic magnetizing current.

6. Each armature winding of a three-phase alternator developes an electromotive force whose fundamental, third and fifth harmonics have maximum values which are as 100 to 25 to 25, respectively. The armature windings are Y-connected to a three-wire line and the neutral junction is grounded. The three-wire line delivers currents to three similar non-inductive receiving circuits which are Y-connected to the line and a voltmeter V is connected between the receiver neutral and the ground. A voltmeter connected across any two of the mains (at the generator) reads 17,850 volts. Find the reading of the voltmeter V. Ans. 2500 volts.

7. Find the reading of a voltmeter connected across one of the armature windings in the previous problem. Ans. 10,606 volts.

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8. The alternator of problem 6 delivers currents to three similar non-inductive receiving circuits which are Y-connected to the generator terminals. Find the value of the quotient actual power delivered *divided by* 3 EI, where E is the voltmeter value of the voltage across each armature winding and I is the ammeter value of the current in each armature winding. Neglect armature reaction and neglect armature resistance. Ans. 0·972.

9. A single-phase alternator (frequency 25 cycles per second) is connected to the home end of a 264-mile two-wire line of negligible wire resistance and negligible leakage, and the distant end of the line is open. Taking unity as the maximum value of the fundamental and of each harmonic of the generator voltage, find the maximum values of the fundamental and of the 3rd, 5th, 7th and 9th harmonics of the voltage across the open end of the line after the line has settled to a steady state of oscillation. Ans. 1·025, 1·275, 2·27, 98·8 and 2·37 respectively.

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PART III

APPENDICES

APPENDIX A

INDUCTANCE AND CAPACITANCE OF LINES

1. Definition of inductance. The most convenient definition of the inductance L of a circuit is that based on the equation

$$\Phi = LI \times 10^8 \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

Where Φ represents the flux linkage caused by the current I. To determine Φ we multiply each element of flux $\Delta\phi$ by the fraction of the total current I which it encircles, and add the results. Thus an amount of flux $\Delta\phi$ which encircles $(1/n)$ of the current I contributes $\Delta\phi/n$ flux linkages to Φ.

The inductance L in equation (i) is expressed in henrys when I is in amperes, and the flux is in maxwells.

2. Inductance per unit length of a two-wire line. If we calculate the flux linkage for a unit length of the line for a given current I, then, according to equation (i) of Art. 1, we have

$$L = (\Phi/I) \times 10^{-8} \quad \ldots\ldots\ldots\ldots\ldots \text{(i)}$$

The flux which links with the two-wires of a line is the flux which passes between the wires (plus the proper fractional parts of the flux elements which crosses through the material of the wires) as shown in Fig. 1. The flux distribution shown in this figure is the flux caused by the current in wire A acting alone superposed upon the flux caused by the current in wire B acting alone, and for purposes of computation it is convenient to consider these components of the total flux separately.

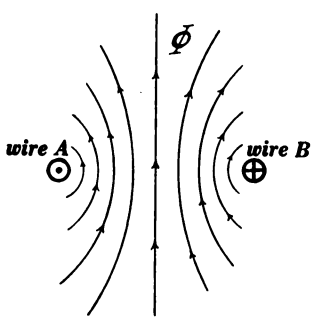

Fig. 1.

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Let us consider the magnetic flux due to the current in wire *A* acting alone. This flux circles around wire *A* in concentric circles, as shown in Fig. 2. Part of it exists inside of wire *A*, part circles around through the space between the two wires, a little bit cuts through wire *B*, and the rest of it circles around both wires. The flux due to the current in wire *B* alone is of an exactly similar character, only the lines of force of this flux are circles concentric with wire *B*. The resultant flux obtained by superimposing the flux surrounding

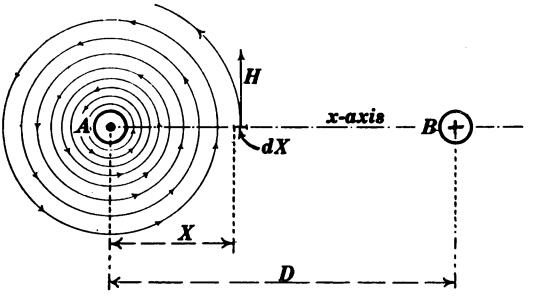

Fig. 2.

wire *A* upon the flux surrounding wire *B* gives the result shown in Fig. 1.

Let us now consider the flux linkage contributed by the flux due to the current in wire *A*; and let us calculate this for one centimeter of line length. The flux which encircles wire *A* can be divided into three parts, namely, the flux inside of wire *A*, the flux passing between wires *A* and *B*, and the flux which encircles both wires.

Calculation of flux linkage due to flux inside of wire A. Consider the flux across the area dx*, as shown in Fig. 3. The flux at this point dx is produced by the fraction (x^2/R^2) of the current I which flows in wire A, so that the flux crossing the area dx is $2\mu I(x^2/R^2)dx/(10x) = \mu dx \cdot Ix/(5R^2)$, where μ is the permeability of the material composing the wire ($\mu = 1$

* We are considering a unit length of the two-wire line so that dx represents an area of width dx and of length unity.

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for non-magnetic materials). This flux links with the fraction (x^2/R^2) of the total current and so represents the following number of flux linkages :

$$d\Phi' = \frac{\mu I x^3}{5R^4} \cdot dx \quad \text{.....(ii)}$$

Integrating this from $x = 0$ to $x = R$ gives the total flux linkage Φ' due to the flux inside of the wire, namely,

$$\Phi' = \frac{\mu I}{20} \quad \text{.....(iii)}$$

Calculation of flux linkage due to flux between wires A and B. Consider the flux across the area dX due to the current in

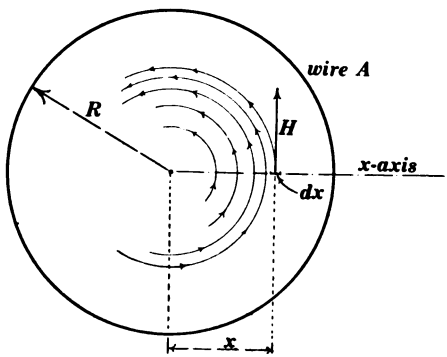

Fig. 3.

wire A, as shown in Fig. 2. The flux density at this point dX is $2I/(10X)$, the flux across dX is $(2I \cdot dX)/(10X)$, and this flux is counted in full since it links with the whole current I, so that

$$d\Phi'' = \frac{I}{5X} \cdot dX \quad \text{.....(iv)}$$

Let us assume that wire B in Fig. 3 is very small,* then there will be no need to consider the flux which crosses through the material of the wire B, and if we integrate (iv) from $X = R$ to $X = D$, we will get an expression for the flux linkages contributed by the flux passing between wires A and B, namely,

$$\Phi'' = \frac{I}{5} \ln\left(\frac{D}{R}\right) \quad \text{.....(v)}$$

* This assumption leads in fact to a correct result.

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Calculation of flux linkage due to flux encircling both wires A and B. This part of the flux encircles $+I$ amperes in wire A, and $-I$ amperes in wire B, so that this part of the flux links with a net current of zero, and so does not contribute any flux linkages.

Evaluation of Φ. The total flux linkage contributed by the current flowing in wire A is $\Phi' + \Phi''$. If we now consider the flux linkage contributed by the current $-I$ flowing in wire B, it is found that the current in wire B contributes the same number of flux linkages as the current in wire A, so that

$$\tfrac{1}{2}\Phi = \left\{\frac{\mu}{20} + \frac{1}{5}\ln\left(\frac{D}{R}\right)\right\} I \text{.....(vi)}$$

Accordingly, the inductance in henrys per centimeter of length is, by (i),

$$L = \left\{\mu + 4\ln\left(\frac{D}{R}\right)\right\} \times 10^{-9} \text{.....(vii)}$$

Since 1 mile = 160,940 cm., and $\ln(D/R) = 2{\cdot}302 \log(D/R)$, the inductance in henrys per mile of two wire line is

$$L = 0{\cdot}0001609 + 0{\cdot}0014822 \log\left(\frac{D}{R}\right) \text{ ....(viii)}$$

3. Inductance per unit length of an ideal two-wire line having zero wire resistance. If the line wires had zero resistance the line current would flow only along the surfaces of the wires, and there would be no internal flux linkages. Therefore for this ideal case the first terms in the second members of equations (vii) and (viii) of Art. 2 are zero. Equations (vii) and (viii) of Art. 2, when the term involving μ is omitted are, however, only approximately true for an ideal resistanceless line. See problem 1 at the end of this Appendix.

4. The average inductance per unit length of a two-wire line with equal-length sections differently spaced. Consider a two-wire line with spacing D' for l cm., with spacing D'' for l cm., and with spacing D''' for l cm. For example, consider *one*

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pair of wires 1 and 2 of the three-wire line which is shown in Fig. 4. Then, using equation (vii) of Art. 2, we get

$$\text{inductance of section } A = l\left[\mu + 4\ln\left(\frac{D'}{R}\right)\right] \times 10^{-9}$$

$$\text{inductance of section } B = l\left[\mu + 4\ln\left(\frac{D''}{R}\right)\right] \times 10^{-9}$$

$$\text{inductance of section } C = l\left[\mu + 4\ln\left(\frac{D'''}{R}\right)\right] \times 10^{-9}$$

and the total inductance of all three sections is the sum of these three expressions; whence, dividing this total inductance

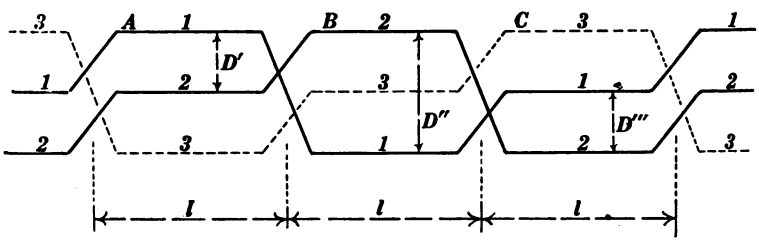

Fig. 4.

by the total length $3l$ of the three sections, we get the average inductance per centimeter of the two-wire line, namely,

$$L = \left\{\mu + 4\ln\left[\frac{(D'D''D''')^{\frac{1}{3}}}{R}\right]\right\} \times 10^{-9} \quad \text{.....(i)}$$

That is to say, the average inductance per mile of a two-wire line with n sections of equal length differently spaced may be calculated by equation (viii) of Art. 2 if for D we use the nth root of the product of the n successive spacings. Equation (i) is derived for $n = 3$.

5. Inductance per unit length of a multi-wire single-phase line. Let us consider, only, the case of a single-phase line consisting of four similar wires, where wires 1 and 2 constitute one conductor and wires 3 and 4 constitute the return conductor. If the four wires are arranged as shown in Fig. 5

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the line current divides equally between the wires of each conductor ($+I/2$ in each of the wires 1 and 2, and $-I/2$ in each of the wires 3 and 4), but this is not the case in Fig. 6 unless the wires of each conductor are transposed at equal distances along the line. A section of the line will be like Fig. 6, in the next section wires 1 and 2 change places and wires 3 and 4 change places, the next section is again like Fig. 6, and so on. A four-wire single-phase line would, in practice, be transposed as stated (if single-phase lines were ever used) because an unequal division of the line current

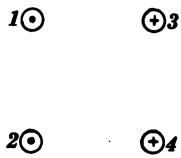

Fig. 5.

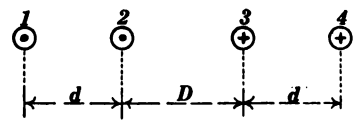

Fig. 6.

Wires 1 and 2 constitute one conductor, and wires 3 and 4 constitute the other conductor.

between the two similar wires of one conductor would give a greater line loss than an equal division of current.

To evaluate the average inductance per unit of a single-phase line like Fig. 5 or like Fig. 6 (transposed as stated), one must find the flux linkage caused by the flux due to the current $I/2$ in wire 1, the flux linkage caused by the flux due to the current $I/2$ in wire 2 (these are in fact equal to each other in Fig. 5, and in Fig. 6 when the wires are transposed as stated), and then add the results to get the total linkage caused by the current I flowing in both wires 1 and 2. This result is to be doubled to get the total linkage Φ because the flux linkage contributed by the current $-I$ flowing in wires 3 and 4 is equal to the flux linkage caused by the current $+I$ flowing in wires 1 and 2.

Let it suffice to derive an expression for the inductance per unit length of the four-wire single-phase line shown in Fig. 6, with the wires transposed as stated. Let us consider the

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transposition sections to be of unit length. Then, using the methods of Art. 2, we find

$$\left\{\begin{matrix}\text{Flux linkage due to flux caused}\\ \text{by current in wire 1, when}\\ \text{wire 1 is located as in Fig. 6}\end{matrix}\right\} =$$

$$\frac{\mu I}{80} + \frac{I}{20}\ln\left(\frac{d}{R}\right) + \frac{I}{10}\ln\left(\frac{D+d}{d}\right) + \frac{I}{20}\ln\left(\frac{D+2d}{D+d}\right) \ldots \text{(i)}$$

$$\left\{\begin{matrix}\text{Flux linkage due to flux caused}\\ \text{by current in wire 1 when wire}\\ \text{1 is located where wire 2 is in}\\ \text{in Fig. 6}\end{matrix}\right\} =$$

$$\frac{\mu I}{80} + \frac{I}{20}\ln\left(\frac{d}{R}\right) + \frac{I}{10}\ln\left(\frac{D}{d}\right) + \frac{I}{20}\ln\left(\frac{D+d}{D}\right) \ldots\ldots \text{(ii)}$$

In these equations, the first term to the right of the sign of equality gives the linkages due to the flux inside of wire 1; the second term gives the linkages due to the flux passing between wires 1 and 2 and linking with $I/2$ amperes; the next term gives the linkages due to the flux passing between wires 2 and 3 (or 4) and encircling $+I$ amperes, while the last term gives the linkages due to the flux passing between wires 3 and 4 and encircling a net current of $+I/2$ amperes. These expressions added together give the flux linkages contributed by the current in one wire in two units of length of the line (in two sections), and there are four wires each contributing the same number of flux linkages. Therefore, following the methods used in Art. 2, we get

$$L' = \left\{\frac{\mu}{2} + \ln\left(\frac{D(d+D)^2(2d+D)}{R^2d^2}\right)\right. \ldots\ldots\ldots \text{(iii)}$$

where L' is in henrys per centimeter of the four-wire single-phase line in Fig. 6 (transposed as stated). If L is the corresponding inductance per mile, we have

$$L = 0{\cdot}00016094\mu + 0{\cdot}0003705\log\left(\frac{D(d+D)^2(2d+D)}{R^2d^2}\right) \text{(iv)}$$

6. Inductance per unit length of any pair of conductors of the 6-wire three-phase line of Figs. 7, 8 and 9. Wires 1

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and 1′ constitute one conductor, wires 2 and 2′ constitute another conductor, and wires 3 and 3′ constitute the third conductor; and the conductors are transposed as shown in Figs. 7, 8 and 9, so that the current in each conductor divides equally between the two wires which constitute the conductor.

It is desired to determine the inductance per unit length of a pair of conductors in Figs. 7, 8 and 9, say the inductance of conductor 1–1′ and conductor 2–2′, assuming the third conductor non-existent. To do this we proceed exactly as in Art. 5. Let R be the radius of each wire; let $+I/2$ be the current in each of the wires 1 and 1′; let $-I/2$ be the current

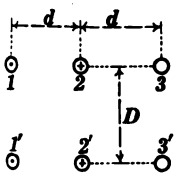

Fig. 7.

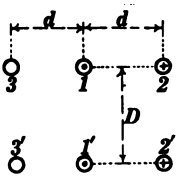

Fig. 8.

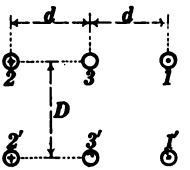

Fig. 9.

in each of the wires 2 and 2′; and let us consider each of the sections to be of unit length. Then

$$\left\{\begin{matrix}\text{Flux linkage caused by flux}\\ \text{due to current in wire 2 or wire}\\ \text{1′ in Fig. 7}\end{matrix}\right\} =$$

$$\frac{\mu I}{80} + \frac{I}{20}\ln\left(\frac{D}{R}\right) + \frac{I}{10}\ln\left(\frac{d}{D}\right) + \frac{I}{20}\ln\left(\frac{d^2+D^2}{d}\right) \ldots\ldots \text{(i)}$$

$$\left\{\begin{matrix}\text{Flux linkage caused by flux due}\\ \text{to current in wire 1 or wire 1′}\\ \text{in Fig. 8}\end{matrix}\right\} = \text{same as (i)} \ldots\ldots\ldots \text{(ii)}$$

$$\left\{\begin{matrix}\text{Flux linkage caused by flux due}\\ \text{to current in wire 1 or wire 1′}\\ \text{in Fig. 9}\end{matrix}\right\} =$$

$$\frac{\mu I}{80} + \frac{I}{20}\ln\left(\frac{D}{R}\right) + \frac{I}{10}\ln\left(\frac{2d}{D}\right) + \frac{I}{20}\ln\frac{\sqrt{4d^2+D^2}}{2d} \ldots\ldots \text{(iii)}$$

In these expressions, the first term to the right of the sign of equality represents the linkages that exist inside of the

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conductor; the next term represents the linkages due to the flux passing between wires 1 and 1′, and linking with $I/2$ amperes; the third term represents the linkages due to the flux passing between wires 1′ and 2 and encircling a current of $+I$; while the last term represents the linkages due to the flux passing between wires 2 and 2′, and encircling a current of $+I$ and $-I/2$, or a net current of $+I/2$.

The sum of equations (i), (ii) and (iii) gives the flux linkages due to the current in one wire for a length of three units. There are four wires each contributing this number of linkages, so that the sum of the equations must be multiplied by (4/3) to get the linkages per unit length. Dividing the linkages per unit length by $I \times 10^8$ gives the inductance in henrys per centimeter length of a pair of conductors in Figs. 7, 8 and 9, namely

$$L = \left\{\frac{\mu}{2} + \tfrac{1}{3}\ln\left(\frac{4d^6\,(d^2+D^2)^2\,(4d^2+D^2)}{R^6 d^6}\right)\right\} \times 10^{-9} \quad \text{..(iv)}$$

7. Inductance per unit length which is to be used for the fictitious two-wire line in Fig. 2 of Chapter IV. Let us consider the fictitious two-wire line in Art. 1 of Chapter IV. The current in this line is exactly the same as the current in one of the wires of the corresponding three-phase line, say wire a of Fig 1, Chapter IV; and the reactive voltage drop per unit length of the fictitious two-wire line is to be exactly equal to the reactive voltage drop in the corresponding single wire a of the three-phase line.

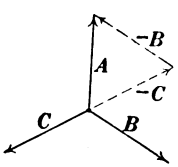

Fig. 10.

Let A, B and C be the vector values of the currents in wires a, b and c, respectively, of the three-phase line as shown in Fig. 10. Since the sum of these three currents is zero, A can be considered as the return circuit for both wires b and c, making the current A in wire a consist of two parts. One part is $-B$, the return for $+B$ flowing in wire b; the other part is $-C$, the return for $+C$ flowing in wire c.

The reactive voltage drop in wire a due to the current

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$-B$ flowing in a is $j\omega L(-B)/2$, where L is the inductance of the two wires a and b. The factor 2 enters in the denominator because the total voltage drop in the one turn made by wires a and b is $j\omega L(-B)$, but only one half of this drop exists in either wire. Similarly, the voltage drop in wire a due to the current component $-C$ is $j\omega L(-C)/2$, where L is the inductance of the two wires a and c (which is the same as the inductance of the two wires a and b). The total voltage drop in wire a is the sum of these component drops, or $j\omega L(-B-C)/2$, or since $A = -B - C$, we get

$$\left\{\begin{matrix}\text{Voltage drop in wire } a \\ \text{of the three-phase line}\end{matrix}\right\} = j\omega \cdot \frac{L}{2} \cdot A$$

Now the voltage drop in the fictitious two-wire line whose inductance is L' is $j\omega L'A$ when the two-wire line carries current A. Therefore if these fwo voltage drops are to be equal we have $L/2 = L'$. That is to say, *the inductance to be used for the fictitious two-wire line of Art.* 1, *Chapter IV is one-half of the inductance of any pair of conductors of the given three-wire three-phase line.* Voltage drop per unit length and inductance per unit length should be used throughout instead of simple voltage drop and inductance.

8. Capacitance per unit length of a two-wire line. If Q is the amount of electric charge in coulombs on unit length of each of the line wires (positive charge on one wire and negative charge on the other wire) and if E is the voltage between the wires, then $C = Q/E$ is the capacitance per unit length of the two-wire line.

The fine radiating lines in Fig. 11 represent the lines of force of the electric field due to wire A alone, *the charge on A being assumed to be uniformly distributed around the wire.* The intensity f of this electric field at a point distant x cm. from the axis of the wire A is $1{\cdot}8 \times 10^{12}Q/x$ volts per cm. (or $2Q/x$ statvolts per cm. if Q is expressed in statcoulombs per cm.). Therefore the voltage drop along dx due to wire A alone is $f{\cdot}dx = 1{\cdot}8 \times 10^{12}(Q/x){\cdot}dx$ volts, and if we integrate

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this expression between $x = R$ and $x = D - R$ we get the part of the voltage between the wires due to the charge on A. This is in fact $\frac{1}{2}E$, where E is the actual voltage between

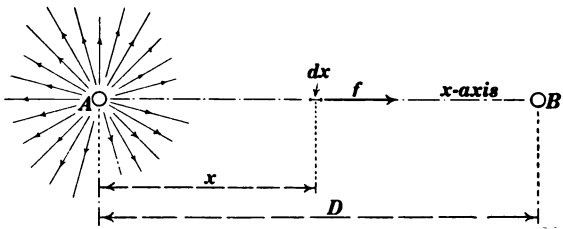

Fig. 11.

A and B are the two line wires each of radius R.

the wires. This is evident because the charge on each wire contributes half of the total voltage. Therefore

$$\tfrac{1}{2}E = \int_{x=R}^{x=D-R} 1{\cdot}8 \times 10^{12}(Q/x){\cdot}dx$$

so that $E = 3{\cdot}6 \times 10^{12} \ln\left(\frac{D-R}{R}\right)$, and, since $C = Q/E$, we have

$$C = \frac{1}{3{\cdot}6 \times 10^{12} \ln\left(\frac{D-R}{R}\right)} \quad \text{.....(i)}$$

This equation gives C in farads per centimeter length of the two-wire line.

The assumption that the charges $+Q$ and $-Q$ are uniformly distributed around the wires gives a value of E which is too small, or a value of $C = Q/E$ which is too large, and if we use D for $D - R$ in equation (i) we will get a value of C which is too small; nevertheless, as exemplified in the note at the end of Art. 10, it is better to use D for $D - R$ so that equation (i) becomes

$$C = \frac{1}{3{\cdot}6 \times 10^{12} \ln\left(\frac{D}{R}\right)} \quad \text{.....(ii)}$$

or, using 1 mile = 160,940 cm., and using $\ln(D/R) = 2{\cdot}302 \log(D/R)$, we get

$$C = \frac{1{\cdot}943 \times 10^{-8}}{\log(D/R)} \text{ farads per mile} \quad \text{.....(iii)}$$

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9. Relation between L and C of an ideal two-wire line having zero resistance. The velocity of electric waves on an ideal line of zero resistance was shown in Chapter I to be equal to $\sqrt{1/(LC)}$; when L is expressed in henrys per mile and C in farads per mile the expression $\sqrt{1/(LC)}$ gives the wave velocity in miles per second; and this for a straight air line is known to be the same as the velocity of light in air, namely, 186,300 miles per second. Therefore

$$LC = 2{\cdot}880 \times 10^{-11} \quad \ldots\ldots\ldots\ldots \text{(i)}$$

so that if either C or L is known the other can be calculated. It must be remembered that the L to be used in this formula does not contain the first term in the second member of equations (vii) and (viii) of Art. 2.

The chief use of the above equation (i) is in the calculation of the proper value of C to use when the value of L has been determined for any complex line whatever or when the proper value of L has been found for the fictitious two-wire line in in Fig. 2, Chapter IV. *To find C omit the term in L which depends on internal flux linkages [the term involving μ in equation (viii) of Art. 2] and use this value of L in equation (i).* Therefore it is unnecessary to develope formulas for the capacitance of complex lines as has been done for inductance in Arts. 4, 5, 6 and 7. The rule given in Art. 1 of Chapter IV that the proper capacitance to use for the fictitious single-phase line in Fig. 2 of Chapter IV is evident, because if the proper inductance to use is $\frac{1}{2}L'$ the proper capacitance must be $2C'$, where L' and C' are the inductance and capacitance per unit length of *any pair* of wires in Fig. 1 of Chapter IV.

This exact relation between C and L (for a resistanceless line) makes it of increased interest to develope the rigorous expression for the capacitance per unit length of a pair of infinitely long parallel metal cylinders.

10. Exact formula for the wire-to-wire capacitance of a two-wire transmission line. The point A in Figs. 12 and 13 represents a geometrical line perpendicular to the plane of the

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paper, and Q coulombs of positive electric charge is supposed to be concentrated on each centimeter of this line. Such a charge is called a *linear charge*, and the lines of force of the electric field due to a linear charge (positive) are radiating straight lines somewhat as shown in Fig 11. The intensity f of the electric field due to the linear charge A at a point distant ρ centimeters from A is

$$f = \frac{1{\cdot}8 \times 10^{12}Q}{\rho} \text{ volts per cm.} \quad \ldots\ldots\ldots\ldots \text{(i)}$$

The equi-potential surfaces in the electric field due to the linear charge A are circular cylinders with their axes along A, and the potential drop* from any point distant ρ cm. from A to any point distant d cm. from A can be found by integrating $f{\cdot}d\rho$ along *any radial line* from ρ to d. Therefore

$$\left\{\begin{array}{l}\text{Potential drop from any point}\\ \text{distant } \rho \text{ from } A \text{ to any point}\\ \text{distant } d \text{ from A}\end{array}\right\} = 1{\cdot}8 \times 10^{12}Q \ln\left(\frac{d}{\rho}\right) \quad \ldots \text{(ii)}$$

Similarly, if B is a linear charge of $-Q$ coulombs per centimeter, we have

$$\left\{\begin{array}{l}\text{Potential drop from any point}\\ \text{distant } \rho' \text{ from B to any point}\\ \text{distant } d \text{ from B}\end{array}\right\} = -1{\cdot}8 \times 10^{12} \ln\left(\frac{d}{\rho'}\right) \quad \text{(iii)}$$

Electric field and potential distribution due to two equal and opposite parallel linear charges. Let A and B, Fig 12, represent the parallel and opposite linear charges. The lines of force of the electric field due to the combined action of A and B are circles passing through A and B as shown by the dotted circles in Fig. 12, and the equi-potential surfaces are circular cylinders whose intersections with the plane of the paper are the full-line circles in Fig. 12.

Let the middle point O in Figs. 12 and 13 be chosen as the origin of co-ordinates and as the region of zero potential (the

* It is much better to speak of *potential drop* than to speak of *potential difference* because the former expression keeps algebraic sign clearly in mind.

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choice of the region of zero potential is always arbitrary). Then the potential V at any point p is the potential drop from p to O, and this potential drop may be considered in two parts, namely, (a) The potential drop from p to O due to charge A alone as given by equation (ii), and (b). The potential

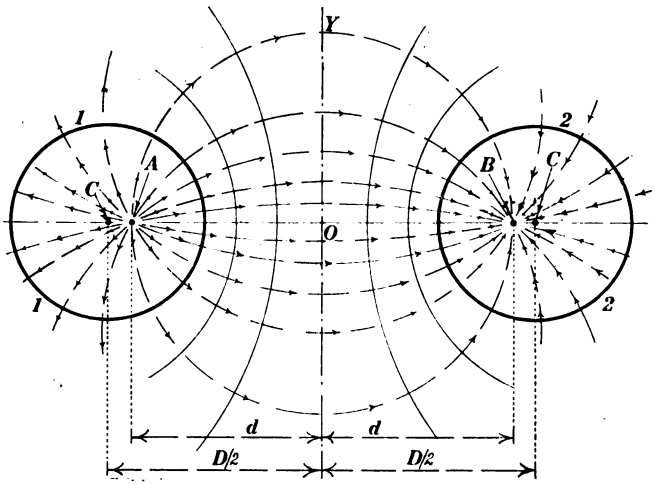

Fig. 12.

Dotted lines represent lines of force.
Full-line circles represent equi-potential surfaces.
Points CC are the centers of the heavy circles 1 and 2.
Distance $AO = d$. Distance $CC = D$.

drop from p to O due to charge B alone as given by equation (iii). Therefore, adding equations (ii) and (iii) we get

$$V = 1{\cdot}8 \times 10^{12} Q \ln \left(\frac{\rho'}{\rho}\right) \quad \ldots\ldots\ldots\ldots \text{(iv)}$$

But from Fig. 13, remembering that the abscissa x as shown is considered as negative, we have $\rho = \sqrt{y^2 + (d + x)^2}$ and $\rho' = \sqrt{y^2 + (d - x)^2}$ so that equation (iv) becomes

$$V = 1{\cdot}8 \times 10^{12} Q \ln \left(\frac{\sqrt{y^2 + (d - x)^2}}{\sqrt{y^2 + (d + x)^2}}\right) \quad \ldots\ldots \text{(v)}$$

but, for brevity, let us write

$$a = \frac{2V}{1{\cdot}8 \times 10^{12} Q} \quad \ldots\ldots\ldots\ldots \text{(vi)}$$

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then equation (v) may be written

$$\frac{y^2 + (d - x)^2}{y^2 + (d + x)^2} = \epsilon^a \quad \ldots\ldots\ldots\ldots \text{(vii)}$$

and equation (vii) is easily reduced to

$$x^2 + y^2 - 2d \cdot \frac{1 + \epsilon^a}{1 - \epsilon^a} \cdot x + d^2 = 0 \quad \ldots\ldots \text{(viii)}$$

Comparing this with the standard form of the equation of a circle whose radius is R and whose center is on the x-axis with an abscissa X, we see that (viii) is the equation of such a circle or rather of a family of such circles for various constant values of a (constant values of V), and it is easily found that

$$R = \frac{2d\epsilon^{a/2}}{1 - \epsilon^a} \ldots\ldots\ldots\ldots\ldots\ldots \text{(ix)}$$

and that

$$X = \frac{D}{2} = \frac{1 + \epsilon^a}{1 - \epsilon^a} \cdot d \quad \ldots\ldots\ldots\ldots \text{(x)}$$

It is desirable to use $D/2$ for the abscissa X as we shall see later. Therefore, substituting the value of d from (x) in equation (ix) we get

$$R = \frac{D\epsilon^{a/2}}{1 + \epsilon^a} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(xi)}$$

Electric field and potential distribution between two parallel metal cylinders at a distance D apart center to center, one cylinder having $+Q$ coulombs of charge per centimeter of length, the other cylinder having $-Q$ coulombs of charge per centimeter of length, and the cylinders having the same radius R. Any equi-potential surface in an electric field might be replaced by a metal (conducting) body of the same size and shape, or any two equi-potential surfaces might be replaced by two such bodies without altering in any way the electric field and potential distribution throughout the intervening dielectric (which is here supposed to be air) if the charge on each metal body is equal to the charge which was originally inside of the equi-potential surface replaced by the body.

The heavy circles in Figs. 12 and 13 represent solid metal cylinders of radius R replacing two of the equi-potential

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surfaces due to the opposite linear charges A and B, and the electric field distribution in the dielectric between the metal cylinders is exactly the same as before if the charges on the cylinders are, respectively, $+Q$ and $-Q$ coulombs per centimeter of length. Therefore equation (v) expresses the potential at any point in the field, the potential at O being taken equal to zero.

The electric field in Fig. 12 is symmetrical with respect to the line OY so that the drop of potential from cylinder 1 to O

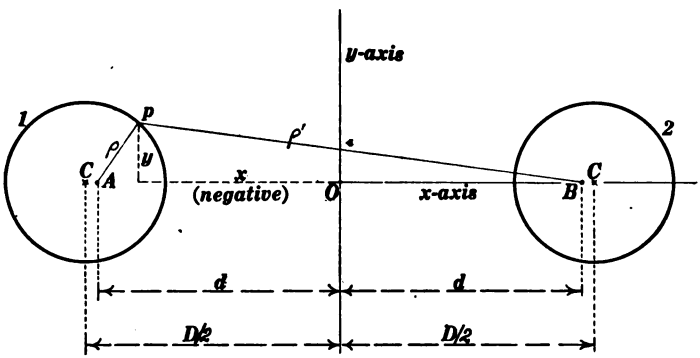

Fig. 13.

Points A and B represent linear charges.
Points C C are the centers of the heavy circles.

is the same as the drop of potential from O to cylinder 2. Therefore if the values of x and y as shown in Fig. 13 be used in equation (v), the value of V will be the potential drop from cylinder 1 to O and the potential difference between the two cylinders 1 and 2 will be $E = 2V$. Therefore, since $C = Q/E$ is the capacitance per unit length of the two cylinders, we have $C = Q/2V$ where V is given by equation (v).

If we use $2V/(1{\cdot}8 \times 10^{12}Q)$ for a in equation (xi) we get an equation containing only V and Q and the given quantities R and D, and if we solve this equation for $Q/2V$ we will have the desired expression for C, namely:

$$C = \frac{1}{3{\cdot}6 \times 10^{12} \ln\left(\frac{D}{2R} \pm \sqrt{\frac{D^2}{4R^2} - 1}\right)} \text{ farads per cm. . . (xii)}$$

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and the positive sign must be chosen for the radical because we know that C becomes very small when D is very large.

Note. This equation (xii) gives $9{\cdot}004 \times 10^{9}$ farads per mile for a two-wire line for which $D = 36$ inches and $R = 0{\cdot}25$ inch; whereas, *for this line*, equation (i) of Art. 8 gives for C a value which is 1 part in 700 too large, and equation (ii) of Art. 8 gives for C a value which is 1 part in 100,000 too small.

PROBLEMS

1. Using equation (xii) of Art. 10 and using equation (i) of Art. 9 in the form which involves c.g.s. units (velocity of light in air = 299,800 kilometers per second) derive an exact expression for the inductance per centimeter length of a two-wire line of zero resistance (no flux inside of the wires).

2. Given a three-wire three-phase line with balanced three-phase currents, spacings D', D'' and D''' radius of wires R, wires being transposed so that each wire occupies each position for $\frac{1}{3}$ of the distance along the line. Set up an expression for the L to be used for the fictitious two-wire single-phase line of Fig 2, Chapter III.

Ans. $L = \frac{1}{2}\mu + 2 \ln \left[\frac{(D'D''D''')^{\frac{1}{3}}}{R} \right]$. Note that this is $\frac{1}{2}$ of the inductance per centimeter of *any pair* of the given line wires as given by equation (i) of Art. 4.

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APPENDIX B

THE COMPLEX-QUANTITY METHOD IN ALTERNATING-CURRENT THEORY

1. Methods in alternating-current theory. The steady-state voltage-current relations in any network of conductors, or in any generator or motor or transformer can be completely represented by the geometric relations of lines (vectors) in a clock diagram if the voltages and currents are harmonic. Therefore any problem in alternating currents which does not involve transient phenomena can be solved in either of two ways, namely, (*a*) Graphically, by drawing the clock diagram vectors carefully to scale so as to represent the given data and measuring off the desired result, or (*b*) Trigonometrically, by establishing the trigonometric equations which express the geometric relations of the clock diagram and solving these equations for the desired results.

The difficulty with the trigonometric method is that the equations are often complicated and the necessary transformations are often very tedious. This difficulty is greatly reduced by the use of *complex quantities* inasmuch as in this case the fundamental equations of a problem can be written down with great ease, especially if a rough sketch of the clock diagram is used as a guide, and because the trigonometric transformations are greatly simplified. It must be remembered, however, that the use of complex quantities in alternating-current theory does not constitute a distinct method; it is merely a beautifully systematic scheme for carrying out method (*b*) above.

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2. The representation of alternating-current quantities by vectors. The student of elementary alternating-current theory is familiar with the clock-diagram method of representing alternating electromotive forces and currents. Any harmonically varying quantity can be completely represented by a vector in a clock diagram. For example, consider a harmonic alternating electromotive force $e = \mathbf{E} \sin \omega t$ and a harmonic alternating current $i = \mathbf{I} \sin (\omega t - \theta)$. Then e and i can be represented by the vectors E and I in Fig. 1. The x-axis is a chosen line from which the angle ωt is measured (the x-axis shows the position of the E vector at $t = 0$).

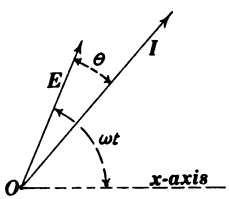

Fig. 1.

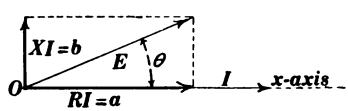

Fig. 2.

3. Representation of vectors. A vector may be specified in two ways; by giving the length or numerical value of the vector and the angle it makes with the x-axis, or by giving the x- and y-components of the vector.

As an example of the first method, consider the vector E in Fig. 2. The numerical value of this vector is e* and it makes an angle θ with the x-axis. Note that the x-axis in Fig. 2 is arbitrarily chosen as coincident with the I vector, but, of course, the x-axis may be chosen in any direction as shown in Fig. 1. The vector of magnitude e making an angle θ with the x-axis is usually written thus, for brevity

$$E = e\underline{/\theta} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

This is called the *polar form* of expression for a vector.

* In elementary alternating-current theory small letters are used for instantaneous values. Thus e and i in Art. 1 represent the instantaneous values of voltage and current. In this text, however, large letters are used for vectors and small letters are used for numerical values. See Art. 1 of Chapter III.

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As an example of the second method, consider the same vector E in Fig. 2. The projection of this vector on the x-axis is a $(= RI)$, the projection on the y-axis is b $(= XI)$; and, if these two projections or components are given the vector E is completely specified. This method of specifying a vector is usually written

$$E = a + jb \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

This is called the *rectangular form* of expression for a vector. The symbol j is here used as a mark to indicate a y-component or projection. The full meaning of the symbol j is discussed in Art. 4.

The relations between the quantities a, b, e and θ can be determined by the geometrical properties of Fig. 2. The fundamental relations are

$$\left.\begin{aligned} e &= \sqrt{a^2 + b^2} \\ \tan\theta &= b/a \\ a &= e\cos\theta \\ b &= e\sin\theta \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \text{(iii)}$$

These equations enable one to transform the polar form (i) into the rectangular form (ii), or vice versa.

4. Meaning of symbol j. In Art. 3 the quantity b is multiplied by j to show that it is a y-component. Any quantity a or b *not* multiplied by j would be understood to be an x-component so that *multiplication by j is interpreted as turning the quantity b 90° forwards from the reference axis.* This is adopted as the full meaning of the symbol j.

Multiplication by j^2 is interpreted as two multiplications by j so that multiplication by j^2 is a forward rotation of 180°. But a forward rotation of 180° merely reverses the direction of a vector and is equivalent to a reversal of algebraic sign, so that $j^2 = -1$. The meanings of j^3, j^4, etc., are found in this way to be $j^3 = -j$, $j^4 = +1$, etc. Multiplication by $-j$ means a backward rotation of 90°.

Since $j^2 = -1$ it is evident that $j = \sqrt{-1}$. If the

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mathematical solution of any electrical problem involves $\sqrt{-1}$ it is permissible to put the symbol j in place of $\sqrt{-1}$.

A term involving j, such as jb, is called an *imaginary quantity* because it introduces j, or $\sqrt{-1}$, although j is not imaginary at all in its meaning, as we have just seen. Terms not involving j are called *real quantities*, and terms, such as $(a + jb)$, which contain both real and imaginary parts, are called *complex quantities*.

5. Impedance as a vector. Figure 2 shows the relation between the voltage E acting on a circuit and the current I in the circuit, the resistance of the circuit being R and its reactance being X; and in Fig. 2 the current vector is chosen as the x-axis of reference. The x-component of E is RI and the y-component of E is XI, so that, using the letter j to mark the y-component, we have

$$E = RI + jXI$$

or

$$E = (R + jX)I \quad \text{.....(i)}$$

The quantity $(R + jX)$ is called the impedance of the circuit which has a resistance R and a reactance X. Impedance is a vector, because it has an x and a y projection, a real and an imaginary part; but impedance does not represent a harmonically varying quantity, and so is not the same kind of a vector as a current or voltage vector, although it has the same properties.

If the impedance vector is denoted by Z, then $Z = R + jX$, and equation (i) can be rewritten

$$E = ZI \quad \text{.....(ii)}$$

Although equation (i) was derived for the special case in which the current vector was the x-axis, equations (i) and (ii) are true even when the current is not the x-axis of reference. The value of E can be determined when Z and I are known, by multiplying the vectors Z and I according to the rules given in Art. 8. When I is unknown, equation (ii) can be rewritten as

$$I = E/Z \quad \text{.....(iii)}$$

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and then I can be found by taking the quotient of the vectors E and Z by the rules given in Art. 9.

6. Admittance as a vector. Figure 3 shows the relation between the voltage E and the current I in a circuit of conductance G and susceptance B, and in this figure the voltage vector is chosen as the x-axis of reference. The x-component

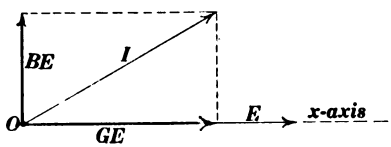

Fig. 3.

of I is GE and the y-component of I is BE, so that, using the letter j to mark the y-component, we have

$$I = GE + jBE$$

or

$$I = (G + jB)E \quad \text{.....(i)}$$

The quantity $(G + jB)$ is called the admittance of the circuit which has a conductance G and a susceptance B. Admittance is a vector because it has an x and a y projection, a real and an imaginary part; but admittance does not represent a harmonically varying quantity, and so is not the same kind of a vector as a current or voltage vector, although it has the same properties.

If the admittance vector is denoted by Y, then $Y = G + jB$, and equation (i) can be rewritten

$$I = YE \quad \text{.....(ii)}$$

Although equation (i) was derived for the special case in which the voltage vector was the x-axis, equations (i) and (ii) are true even when the voltage is not the x-axis of reference. When two of the vectors in equation (ii) are known, the third one can always be found, either by vector multiplication as explained in Art. 8, or by vector division, as explained in Art. 9.

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7. Addition and subtraction of vectors. In order to be added or subtracted, vectors must first be expressed in rectangular form. After this has been done the general rule for addition is that the x-projection of the sum is the sum of the x-projections of the vectors to be added, and the y-projection of the sum is the sum of the y-projections of the vectors to be added.

Similarly, the x-projection of the difference of two vectors is the difference of the x-projections of the vectors being subtracted, and the y-projection of the difference is the difference of the y-projections of the vectors being subtracted.

As an example of vector addition, let us add the vectors $E = (a + jb)$, and $E' = (a' + jb')$. Then the sum is $E + E' = (a + a') + j(b + b')$. As an example of vector subtraction, let us subtract the same vectors. Then $E - E' = (a - a') + j(b - b')$.

8. Multiplication of vectors. As explained in Arts. 5 and 6, it is frequently necessary to multiply vectors. This multiplication can be carried out when the vectors are expressed either in polar form or rectangular form, and the choice of the form to be used in any special case is one of convenience.

When multiplication is carried out with the vectors in the polar form, the length of the vector product is the product of the lengths of the vectors being multiplied, and the angle of the vector product is the sum of the angles of the vectors being multiplied. Thus consider equation (ii) of Art. 5, in which $I = i\underline{/\theta'}$, and $Z = z\underline{/\psi}$. Then according to the rule just given, $E = zi\underline{/\theta' + \psi}$.

When the vectors are expressed in rectangular form, the multiplication is carried out by the ordinary rules of algebra, remembering that $j^2 = -1$, $j^3 = -j$, etc. Thus, if $I = (c + jd)$ and $Z = (R + jX)$, then

$$E = ZI = (R + jX)(c + jd)$$

By the ordinary rules of algebra this gives

$$E = Rc + j^2dX + j(Xc + Rd)$$

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and replacing j^2 by -1 gives the final result

$$E = (Rc - Xd) + j(Xc + Rd)$$

This result is exactly the same vector that was obtained by the polar method of multiplication.

9. Division of vectors (and the removal of imaginary quantities in a denominator). As explained in Arts. 5 and 6, it is frequently necessary to divide vectors. This division can be carried out when the vectors are expressed either in polar form or rectangular form, and the choice of the form to be used in any particular case is one of convenience.

When division is carried out with vectors in the polar form, the length of the product is the quotient of the lengths of the vectors being divided, and the angle of the quotient is the difference of the angles of the vectors being divided. Thus consider equation (iii) of Art. 5, in which $E = e\underline{/\theta}$, and $Z = z\underline{/\psi}$. Then, according to the rule just given, $I = (E/Z) = (e/z)\underline{/\theta - \psi}$.

When division is carried out with vectors in the rectangular form, imaginary quantities ordinarily appear in the denominator of a fraction, and it is necessary to remove these imaginary quantities from the denominator in order to make the result understandable. Thus, if $E = a + jb$, and $Z = R + jX$, then $I = E/Z = (a + jb)/(R + jX)$, and this quotient is meaningless as it stands. To remove the j from the denominator, one multiplies both numerator and denominator by a binomial which is the same as the denominator, but in which the sign of the j term has been reversed. In the example being considered, one accordingly multiplies both numerator and denominator of the fraction by $(R - jX)$. Doing this gives, by the rules of multiplication

$$I = \frac{a + jb}{R + jX} \times \frac{R - jX}{R - jX} = \frac{(aR + bX)}{R^2 + X^2} + \frac{j(bR - aX)}{R^2 + X^2}$$

This result gives exactly the same vector that was obtained by division using the polar method.

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10. Powers and roots of vectors. When it is desired to raise a vector quantity to some power, or to take some root of a vector quantity, one must first express the given vectors in polar form. The principle that governs when a vector is raised to the nth power is that the length of the resultant vector is the nth power of the length of the given vector, and that the angle of the resultant vector is n times the angle of the given vector. Accordingly, the nth power of the vector $Z = z\underline{/\psi}$ is $z^n\underline{/n\psi}$.

Similarly, when the nth root of a vector is taken, this is equivalent to raising the vector to the $(1/n)$th power, and the length of the resultant vector is the nth root of the length of the given vector, and the angle of the resultant vector is $1/n$th of the angle of the given vector. Accordingly the nth root of the vector Z is $z\underline{/\psi}\ \sqrt[n]{z}\underline{/\psi/n}$.

11. Relation of admittance to impedance. The meaning of the admittance and impedance vectors was discussed in Arts. 5 and 6. According to equation (iii) of Art. 5, $I = E/Z$, and according to equation (ii) of Art. 6, $I = YE$. It is thus evident that the admittance must be the reciprocal of the impedance, and if one is known it is possible to determine the other. The operation of taking the reciprocal of a vector is merely one form of division, so that the transformation from admittance to impedance vectors can be made without introducing any new principles. As an example, let us find the admittance Y when $Z = R + jX$. Then $Y = 1/(R + jX)$, and multiplying both numerator and denominator by $(R - jX)$ in order to remove the imaginaries from the denominator, as described in Art. 9, gives

$$Y = \frac{1}{R + jX} \times \frac{R - jX}{R - jX} = \frac{R}{R^2 + X^2} - j\frac{X}{R^2 + X^2}$$

accordingly, since $Y = G + jB$, it is evident that

$$G = \frac{R}{R^2 + X^2}$$

and

$$B = -\frac{X}{R^2 + X^2}$$

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Note that susceptance B is negative when reactance X is positive.

12. De Moivre's theorem. The use of complex quantities in alternating-current theory is very greatly facilitated by making use of the relation

$$\epsilon^{j\theta} = \cos\theta + j\sin\theta \quad \ldots\ldots\ldots\ldots\ldots \text{(i)}$$

where ϵ is the Naperian base ($= 2{\cdot}7183$). This relation is due to De Moivre and it can be established by putting $j\theta$ for x in the infinite series for ϵ^x $\left(= 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} + \text{etc.}\right)$ and separating the terms which do not contain j as a factor from the terms which do contain j as a factor. The former group of terms is the series for $\cos\theta$ and the latter group of terms is the j times the series for $\sin\theta$.

13. Proposition. *Multiplying any vector by* $\epsilon^{j\theta}$ *or by* $\cos\theta + j\sin\theta$ *turns the vector forwards* (*in the conventional direction of clock-diagram rotation*) *through the angle* θ.

This proposition is true for any vector whatever, but we will prove it only for the simple case in which the given vector is in the direction of the x-axis and therefore wholly real (no y-component). Let I be the given vector. Multiplying by $\cos\theta + j\sin\theta$ we get

$$I\epsilon^{j\theta} = I\cos\theta + jI\sin\theta$$

so that the x-component of $I\epsilon^{j\theta}$ is $I\cos\theta$ and the y-component of $I\epsilon^{j\theta}$ is $I\sin\theta$, and it is evident, therefore, that $I\epsilon^{j\theta}$ is a vector whose length is I and which makes an angle θ with the x-axis.

Remark. Dividing a given vector by $\epsilon^{j\theta}$ (or multiplying the given vector by $\epsilon^{-j\theta}$) turns the vector backwards through the angle θ.

14. Coils in series. To illustrate the use of complex quantities in alternating-current theory let us consider the problem of two coils connected in series across alternating-current supply mains as shown in Fig. 4, E, r_1, r_2, x_1 and

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x_2 being given. Make a sketch of the clock diagram as shown in Fig. 5, where the vector I represents the current flowing through the two coils. Then we have, at once, the following equations

$$E_1 = (r_1 + jx_1)I \quad \ldots\ldots\ldots\ldots\ldots\text{(i)}$$

$$E_2 = (r_2 + jx_2)I \quad \ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

and

$$E = E_1 + E_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(iii)}$$

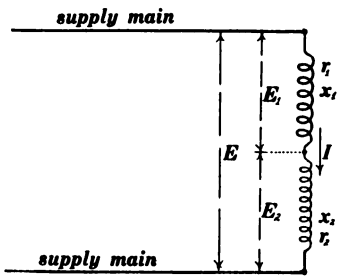

Fig. 4.

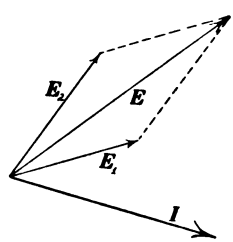

Fig. 5.

Substituting the values of E_1 and E_2 from (i) and (ii) in (iii) and solving for I, we have:

$$I = \frac{E}{r_1 + r_2 + j(x_1 + x_2)} \quad \ldots\ldots\ldots\text{(iv)}$$

Substituting this value in (i) and (ii) we have:

$$E_1 = \frac{(r_1 + jx_1)E}{r_1 + r_2 + j(x_1 + x_2)} \quad \ldots\ldots\ldots\text{(v)}$$

and

$$E_2 = \frac{(r_2 + jx_2)E}{r_1 + r_2 + j(x_1 + x_2)} \quad \ldots\ldots\ldots\text{(vi)}$$

These equations (v) and (vi) express the electromotive forces E_1 and E_2 in terms of the known quantities E, r_1, r_2, x_1 and x_2.

In order to make actual numerical calculations, of E_1 for example, the second member of (v) must be separated into

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its components, that is, into real and imaginary parts, and then the numerical value of E_1 is found by taking the square root of the sum of the squares of these components; and the tangent of the angle between E and E_1 (the angle by which E_1 leads E in phase) is equal to the j-component of E_1 divided by the real component of E_1.

Thus, multiplying numerator and denominator of (v) by $[r_1 + r_2 - j(x_1 + x_2)]$, j is removed from the denominator, and we have:

$$E_1 = E\frac{r_1(r_1+r_2)+x_1(x_1+x_2)}{(r_1+r_2)^2+(x_1+x_2)^2} + jE\frac{x_1(r_1+r_2)-r_1(x_1+x_2)}{(r_1+r_2)^2+(x_1+x_2)^2} \quad \ldots\text{(vii)}$$

The first term of this expression is the component of E_1 parallel to E and the second term, dropping j, is the component of E_1 perpendicular to E. Taking the square root of the sum of the squares of these components we have:

$$\textit{Numerical value of } E_1 = \frac{\sqrt{r_1^2+x_1^2}}{\sqrt{(r_1+r_2)^2+(x_1+x_2)^2}} \times E \quad \ldots\text{(viii)}$$

Taking the ratio of the two components we have:

$$\textit{Tangent of angle from } E \textit{ forwards to } E_1 = \frac{x_1(r_1+r_2)-r_1(x_1+x_2)}{r_1(r_1+r_2)+x_1(x_1+x_2)} \text{(ix)}$$

A negative value of this tangent indicates that E_1 is behind E in phase.

15. General outline of the method of using complex quantities in the solution of a problem in alternating currents. The above discussion illustrates the general use of complex quantities in alternating-current problems, and the procedure is always essentially the same, namely: *First*, the formulation of the problem by equations expressing the relation between current and voltage in elementary circuits, and by equations expressing the known sum of certain voltages when elementary circuits are in series, or the known sum of certain currents when elementary circuits are in parallel; *Second*, the transformation of these complex equations so as to express the

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unknown quantities in terms of their components as in equation (vii) Art. 14; *Third*, the derivation of equations (not involving complex quantities) expressing numerical values and phase angles in terms of the given data; and, *Fourth*, the actual calculation of numerical results.

A fifth step is necessary when it is desired to calculate power which is equal to the numerical value (effective) of voltage times the numerical value (effective) of current times the cosine of phase difference. When current and voltage are found in terms of their components, for example, when:

$$I = i_1 + ji_{11}$$

and

$$E = e_1 + je_{11}$$

then the power may be easily calculated by the formula:

$$P = e_1 i_1 + e_{11} i_{11}$$

This equation does not of course involve j in any way, but in using this equation especial care must be taken to give to each component its proper algebraic sign.

16. Coils in parallel. A given alternating current I divides between two coils connected in parallel as shown in Fig. 6.

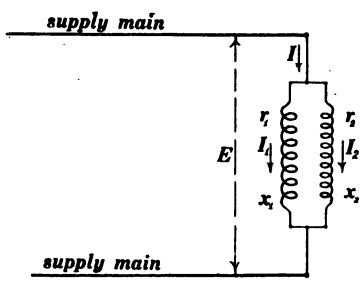

Fig. 6.

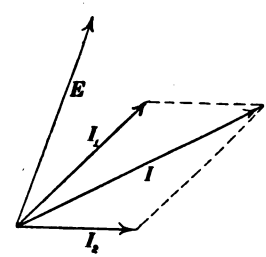

Fig. 7.

It is required to find I_1 and I_2, each in terms of I, r_1, r_2, x_1 and x_2. The general relation between I, I_1, I_2, and the voltage E between the branch points is shown in the clock diagram Fig. 7. The equations are somewhat simplified

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if we introduce the conductance g and the susceptance b of each according to the results given at the end of Art. 11, namely,

$$g_1=\frac{r_1}{r_1^2+x_1^2},\quad b_1=-\frac{x_1}{r_1^2+x_1^2},\quad g_2=\frac{r_2}{r_2^2+x_2^2} \text{ and } b_2=-\frac{x_2}{r_2^2+x_2^2}$$

Then

$$I_1 = (g_1 + jb_1)E \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

$$I_2 = (g_2 + jb_2)E \quad \ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

and

$$I_1 + I_2 = I \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(iii)}$$

The problem is solved from these three equations in a manner similar to the solution of the problem of coils in series in Art. 14.

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APPENDIX C

ELECTRICAL TRANSIENTS IN CIRCUITS CONTAINING ONLY LUMPED INDUCTANCE AND LUMPED CAPACITY

1. Lumped inductance and lumped capacity. Figure 1 shows a heavy ball fixed to the end of a flat spring which is clamped in a vice. In formulating the motion of such a system the mass of the spring may be neglected and the elasticity of the ball may be neglected, that is to say, it is allowable to treat the ball-and-spring as if all of the mass were in the ball and all the elasticity in the spring. The ball and spring constitute a system in which mass and elasticity are *lumped.*

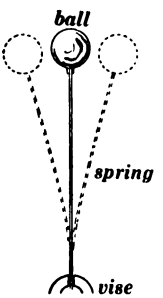

Fig. 1.
Lumped mass and lumped elasticity.

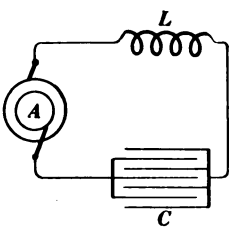

Fig. 2.
Lumped inductance and lumped capacity.

On the other hand, to formulate the motion of the spring alone in Fig. 3 it would be necessary to take into consideration the mass and the elasticity of every part of the spring; the spring alone is an example of a mechanical system in which mass and elasticity are *distributed* or *mixed.*

Figure 2 shows an alternator A delivering current to a circuit consisting of an inductance coil L and a condenser C

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connected in series. In formulating the voltage-current relations in Fig. 2 it is allowable to neglect the minute amount of inductance that is associated with the currents which spread the electric charges over the plates of the condenser and it is allowable to neglect the condenser action of adjacent wires in the inductance coil. Every portion of the connecting wires in Fig. 2 has both inductance and capacity and it is allowable to neglect this inductance and capacity also. The coil and condenser in Fig. 2 constitute a system in which inductance and capacity are *lumped*. On the other hand the

Fig. 3.
Mixed mass and elasticity.

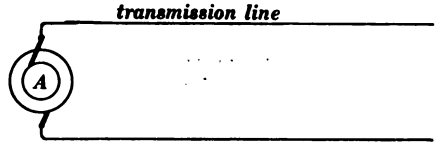

Fig. 4.
Mixed inductance and capacity.

transmission line in Fig. 4 constitutes a system in which inductance and capacity are *distributed* or *mixed*. Every element of the transmission line has both inductance and capacity.

Mechanical systems containing mixed mass and elasticity and electrical circuits containing mixed inductance and capacity exhibit the phenomena of wave propagation (see Chapters I, II and III and see Arts. 1–9 of Chapter VI).

Circuits containing only lumped inductances and capacities (and mechanical systems containing only lumped masses and elasticities) do not exhibit the phenomena of genuine wave propagation. The wave filter, for example, as discussed in Chapter V, exhibits only a kind of pseudo-wave propagation.

The mathematical theory of any mechanism in which weights and springs are linked together can be built up from the mathematical solution of the simple system shown in Fig. 1,

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and the mathematical theory of any network of circuits containing only lumped inductances and capacities can be built up from the solution of the simple circuit shown in Fig. 2. Thus the theory of the wave filter as developed in Chapter V is developed from the solution of the simple circuit shown in Fig. 2.

2. Steady states and transients. Consider any electrical network. The " external influence " under which the network is placed may be specified by giving the voltages (d.c. or a.c.) which are impressed across the various parts of the network and by giving the currents (d.c. or a.c.) which enter or leave the network at various points.

If the " external influence " is steady* the network will quickly settle to a steady state. The definition of the steady state which is given in Chapter III is applicable to any network, and this definition can be easily altered to suit the direct-current case.

The settling of a network to a steady state is called a *transient*; any change of " external influence " or any change in the network itself gives rise to a *transient* which exists while the network is passing from the old steady state to a new steady state. Thus the opening or closing of a switch, the adjustment of a rheostat, the change of an inductance, the change of a capacity, etc., give rise to transients.

Transients are ignored in many practical problems because a transient is usually of very short duration, whereas the ensuing steady state lasts indefinitely and is therefore highly important. When computing steady state conditions, however, it is well to keep in mind the fact that the computations are valid only after the transient which establishes the steady state has run its course.

Many characteristic transients on a transmission line are discussed in Chapter I. See especially problems 35–40 of

* The reader will readily understand what is meant without a full and precise statement.

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Chapter I. No attempt is made in this volume to formulate transmission line transients for the case in which the line losses are appreciable.

This Appendix is devoted to the discussion of transients in circuits containing only lumped inductances (coils) and lumped capacitances (condensers). Every such circuit, however, contains some distributed or mixed inductance and capacity, but the effects of this mixed inductance and capacity is ignored in this Appendix.

3. Current-voltage relations for lumped resistance, lumped inductance and lumped capacity. Let i be the instantaneous value of a current flowing through a circuit, then

(*a*) The instantaneous value of the voltage across a lumped resistance R is

$$e = Ri \quad \text{.....(i)}$$

It is not necessary, however, for resistance to be lumped, and the general interpretation of equation (i) is that Ri is **that portion** of the voltage acting across any part of a circuit which is being used to overcome the resistance of that part of the circuit.

(*b*) The instantaneous value of the voltage across a lumped inductance L is

$$e = L\frac{di}{dt} \quad \text{.....(ii)}$$

This voltage causes the current to grow. If the lumped inductance has resistance R, then the total voltage across L and R is

$$e = Ri + L\frac{di}{dt} \quad \text{.....(iii)}$$

(*c*) If q is the charge on a lumped capacity C (positive charge on one plate of a condenser and negative charge on the other plate) then the voltage across the lumped capacity is

$$e = \frac{q}{C} \quad \text{.....(iv)}$$

and if the lumped capacity has leads of resistance R the total voltage across C and R is

$$e = R\frac{dq}{dt} + \frac{q}{C} \quad \text{.....(v)}$$

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This relation can be expressed in terms of current by placing i for $\frac{dq}{dt}$ and $\int i \cdot dt$ for q, giving

$$e = Ri + \frac{1}{C}\int i \cdot dt \quad \ldots\ldots\ldots\ldots\ldots \text{(vi)}$$

(*d*) The voltage across the terminals of a circuit containing resistance R, lumped inductance L and lumped capacity C, all in series, is

$$e = L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C} \cdot q \quad \ldots\ldots\ldots \text{(vii)}$$

4. General solution of equation (vii) of Art. 3 when e is zero. When $e = 0$ equation (vii) of Art. 3 becomes

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0 \quad \ldots\ldots\ldots\ldots \text{(i)}$$

and the general solution of this differential equation can be set up by the method exemplified in Art. 4 of Chapter I. That is, we may put

$$q = A\epsilon^{at} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

where A and a are constants. Then

$$\frac{dq}{dt} = aA\epsilon^{at} \quad \text{and} \quad \frac{d^2q}{dt^2} = a^2A\epsilon^{at}$$

and if we substitute these values of q, dq/dt and d^2q/dt^2 in equation (i) we get

$$La^2 + Ra + 1/C = 0 \quad \ldots\ldots\ldots\ldots \text{(iii)}$$

Solving this for a we get two values for a, namely,

$$a_1 = -\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} \quad \ldots\ldots \text{(iv)}$$

and

$$a_2 = -\frac{R}{2L} - \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} \quad \ldots\ldots\ldots \text{(v)}$$

Therefore (ii) gives two independent solutions of (i) if a_1 and a_2 are not equal, namely, $q = A_1\epsilon^{a_1t}$ and $q = A_2\epsilon^{a_2t}$, and the sum of these is also a solution of (i). That is to say

$$q = A_1\epsilon^{a_1t} + A_2\epsilon^{a_2t} \quad \ldots\ldots\ldots\ldots\ldots \text{(vi)}$$

is a solution of (i), it is in fact the general solution of (i) because it contains the two constants of integration A_1 and A_2.

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When a_1 and a_2 are equal (when $R^2/4L^2 = 1/LC$), equation (vi) degenerates to an expression involving but one undetermined constant, namely, $A_1 + A_2$, and in this case the general solution of (i) is found to be

$$q = B_1\epsilon^{at} + B_2 t\epsilon^{at} \quad \ldots\ldots\ldots\ldots\ldots \text{(vii)}$$

where $a = -R/2L$ and where B_1 and B_2 are undetermined constants.

Example. A condenser of capacity C has an initial charge Q and at $t = 0$ it is connected to and allowed to discharge through an inductance L and a resistance R connected in series. The varying value of the charge q on the condenser is then expressed as a function of elapsed time by equation (vi) if a_1 and a_2 are not equal to each other or by equation (vii) if $a_1 = a_2 = a$, and the integration constants must be determined from the given initial conditions.

The initial conditions are

$$q = Q \text{ when } t = 0 \text{, and}$$
$$i = \frac{dq}{dt} = 0 \text{ when } t = 0 .$$

Therefore if equation (vi) is the general solution we get

$$\left.\begin{aligned} Q &= A_1 + A_2 \\ \text{and} \quad 0 &= a_1A_1 + a_2A_2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \text{(viii)}$$

from which to calculate the values of A_1 and A_2, whereas if equation (vii) is the general solution we get

$$\left.\begin{aligned} Q &= B_1 \\ \text{and} \quad 0 &= aB_1 + B_2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots \text{(ix)}$$

from which to calculate the values of B_1 and B_2.

Remark. There are three distinct cases of the above problem, namely,

Case I when $R^2/4L^2$ is greater than $1/LC$,
Case II when $R^2/4L^2$ is equal to $1/LC$, and
Case III when $R^1/4L^2$ is less than $1/LC$.

and before describing these three cases it is advisable to transform equations (vi) and (vii) so as to express the value of the current i in the circuit. Therefore, putting $i = \frac{dq}{dt}$

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equations (vi) and (vii) give

$$i = F_1\epsilon^{a_1 t} + F_2\epsilon^{a_2 t} \quad \ldots\ldots\ldots\ldots\ldots (x)$$

and

$$i = G_1\epsilon^{at} + G_2 t\epsilon^{at} \quad \ldots\ldots\ldots\ldots\ldots (xi)$$

where F_1, F_2, G_1 and G_2 are constants of integration.

Futhermore let us describe cases I, II and III for the following initial conditions, namely, a current i_0 is supposed to be flowing at $t = 0$ in a circuit containing R, L and C in series and the charge on the condenser is supposed to be zero at $t = 0$. This second condition means that at $t = 0$ the Ri drop, only, is causing the current to decrease so that at $t = 0$ we have the two conditions*

$$\left.\begin{aligned} i &= i_0 \text{ when } t = 0 \text{, and} \\ L\frac{di}{dt} &= -Ri_0 \text{ when } t = 0 \end{aligned}\right\} \quad \ldots (xii)$$

Case I. In this case a_1 and a_2 are unequal, they are both purely real, and both negative. Therefore the decreasing current in the circuit is expressed by the sum of the two decaying exponential terms in equation (x). The decreasing current is shown graphically in Fig. 5 for this case.

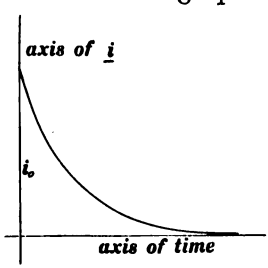

Fig. 5.
Decreasing current for case I.

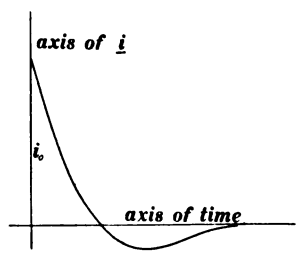

Fig. 6.
Decreasing current for case II.

Case II. In this case $a_1 = a_2 = -R/2L$ and the decreasing current is shown graphically in Fig. 6.

Case III. In this case a_1 and a_2 are complex, in fact

$$a_1 = -\alpha + j\beta \qquad \text{and} \qquad a_2 = -\alpha - j\beta$$

where $\alpha = R/2L$ and $\beta = +\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$

* Let the reader write out the equations which determine F_1 and F_2, and the equations which determine G_1 and G_2.

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It is to be noted that α and β as here used are entirely different from α and β as used in Chapter III. Using the above expressions for a_1 and a_2, equation (x) becomes

$$i = \epsilon^{-\alpha t}(F_1\epsilon^{+j\beta t} + F_2\epsilon^{-j\beta t})$$

which, by De Moivre's theorem, as stated in Art. 6 of Appendix B, reduces to

$$i = \epsilon^{-\alpha t}[F_1(\cos \beta t + j \sin \beta t) + F_2(\cos \beta t - j \sin \beta t)] \quad \text{(xiii)}$$

Now conditions (xii) as applied to equation (x) give

$$F_1 = -\frac{\alpha - j\beta}{2j\beta} \cdot i_0 \quad \ldots\ldots\ldots\ldots \text{(xiv)}$$

and

$$F_2 = \frac{\alpha + j\beta}{2j\beta} \cdot i_0 \quad \ldots\ldots\ldots\ldots\ldots \text{(xv)}$$

and, if we substitute these values in (xiii), separate the whole into real and imaginary parts, and drop the imaginary part, we get

$$i = i_0\epsilon^{-\alpha t}\left(\cos \beta t - \frac{\alpha}{\beta} \cdot \sin \beta t\right)$$

and this is easily reduced to

$$i = i_0'\epsilon^{-\alpha t} \sin (\beta t + \phi) \quad \ldots\ldots \text{(xvi)}$$

where

$$i_0' = i_0\sqrt{1 + \frac{\alpha^2}{\beta^2}} \quad \ldots\ldots\ldots\ldots \text{(xvii)}$$

and

$$\tan \phi = -\frac{\beta}{\alpha} \quad \ldots\ldots\ldots\ldots\ldots \text{(xviii)}$$

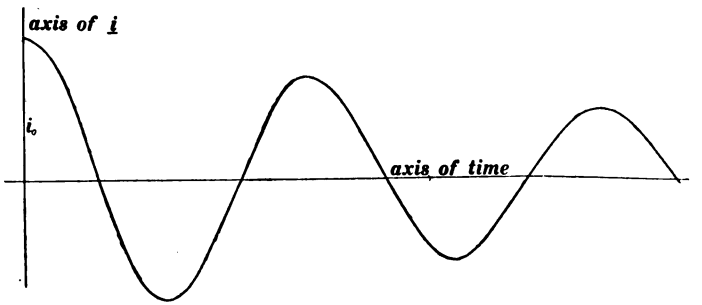

Fig. 7.
Decaying oscillatory current.

The value of i as given by equation (xvi) is shown in Fig. 7.

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5. Steady-state solution and transient or general solution of equation (vii) of Art. 3 when e is constant and equal to E. In this case equation (vii) of Art. 3 becomes

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + q/C = E \quad \text{.....(i)}$$

and it is convenient to rewrite for reference the form of equation (vii) of Art. 3 when e is zero, namely

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + q/C = 0 \quad \text{.....(ii)}$$

Proposition. Let q' be a solution of (i) and let q'' be a solution of (ii). Then $q' + q''$ is a solution of (i).

Proof. Since q' and q'' are solutions of (i) and (ii) respectively, we have

$$L\frac{d^2q'}{dt^2} + R\frac{dq'}{dt} + q'/C = E$$

and

$$L\frac{d^2q''}{dt^2} + R\frac{dq''}{dt} + q''/C = 0$$

adding these equations and collecting terms, we get

$$L\frac{d^2(q' + q'')}{dt^2} + R\frac{d(q' + q'')}{dt} + (q' + q'')/C = E$$

which shows that $q' + q''$ is a solution of (i).

An important consequence of this proposition is that if we can find any solution of (i), however simple and special, then we can get the general solution of (i) by adding the general solution of (ii) to this simple special solution of (i), because the result will be a solution of (i) containing two constants of integration.

The simplest solution of (i) is $q = CE$. This is in fact the steady-state solution of (i), and it is a special solution because it contains no undetermined constants.

The general solution of (i) may now be found by adding equation (vi) of Art. 4 to $q = CE$ which gives

$$q = CE + A_1\epsilon^{a_1t} + A_2\epsilon^{a_2t} \quad \text{.....(iii)}$$

which is the general solution of (i) when equation (iii) of

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Art. 4 has unequal roots; or by adding equation (vii) of Art. 4 to $q = CE$ which gives

$$q = CE + B_1\epsilon^{at} + B_2 t\epsilon^{at} \quad \ldots\ldots\ldots\ldots\text{(iv)}$$

which is the general solution of equation (i) when equation (iii) of Art. 4 has equal roots.

6. Steady-state solution and transient or general solution of equation (vii) of Art. 3 when e is a harmonic alternating voltage. The expression for a harmonic alternating voltage is

$$e = \mathbf{E} \sin \omega t$$

where $\mathbf{E}$ is a constant and $\omega = 2\pi f$ where f is the frequency.

When e is a harmonic alternating voltage equation (vii) of Art. 3 becomes

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + q/C = \mathbf{E} \sin \omega t \quad \ldots\ldots\ldots\text{(i)}$$

and it is convenient to rewrite for reference the form of equation (vii) of Art. 3 when $e = 0$, namely

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + q/C = 0 \quad \ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

Proposition. Let q' be a solution of (i) and let q'' be a solution of (ii). Then $q' + q''$ is a solution of (i). The proof of this proposition is nearly identical to the proof of the corresponding proposition in Art. 5.

An important consequence of this proposition is as follows: Having obtained any solution of (i), however simple and special, we can get the general solution of (i) by adding the general solution of (ii) to this special solution of (i).

The simplest solution of (i) is the steady state solution

$$q = \mathbf{Q} \sin (\omega t - \theta') \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(iii)}$$

where $\mathbf{Q}$ is the maximum charge on the condenser, and where the instant of maximum charge on the condenser comes θ' electrical degrees after the instant at which the applied alternating-current voltage is maximum.

The values of $\mathbf{Q}$ and θ' can be obtained by applying the ordinary alternating-current theory to a circuit containing

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R, L and C in series, keeping in mind that the maximum voltage across the condenser is equal to $\mathbf{Q}/C$. Doing this gives

$$\mathbf{Q} = \frac{\mathbf{E}}{\omega\,[R^2 + (\omega L - 1/\omega C)^2]^{\frac{1}{2}}} \quad \ldots\ldots\ldots \text{(iv)}a$$

and

$$\tan\theta' = \frac{R}{-\omega L + 1/C\omega} \quad \ldots\ldots\ldots\ldots\ldots \text{(v)}a$$

If $\mathbf{I}$ is the maximum value of the alternating current that flows in the circuit, and θ is the angle by which this current lags behind the applied voltage, then the ordinary alternating-current theory gives

$$\mathbf{I} = \mathbf{E}/\sqrt{R^2 + (\omega L - 1/\omega C)^2} \quad \ldots\ldots\ldots \text{(iv)}b$$

$$\mathbf{I} = \omega\mathbf{Q} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(iv)}c$$

$$\tan\theta = \frac{\omega L - 1/\omega C}{R} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(v)}b$$

The vector diagram of these equations is shown in Fig. 8.

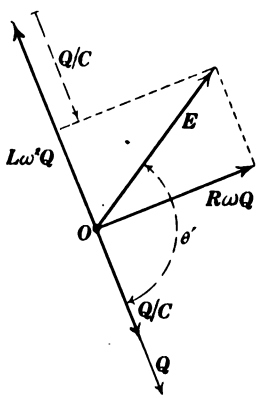

Fig. 8.

A condenser, an inductance, and a resistance are connected in series to a.c. supply mains (a.c. supply voltage harmonic). The vector $\mathbf{E}$ represents the supply voltage, and this voltage causes a current $\mathbf{I}$ ($= \omega\mathbf{Q}$) to flow which lags θ degrees behind $\mathbf{E}$ and produces a voltage drop in the resistance shown by the vector $R\omega\mathbf{Q}$. The voltage across the condenser lags 90° behind $R\omega\mathbf{Q}$, or θ' degrees behind $\mathbf{E}$, and is the vector $\mathbf{Q}/C$, while the charge on the condenser is in phase with the condenser voltage, and is given by the vector $\mathbf{Q}$. The voltage drop across the inductance leads the resistance drop by 90°, and has a value of $L\omega^2\mathbf{Q}$ ($= L\omega\mathbf{I}$), as shown in the figure.

General solution of equation (i). According to the above proposition we must add the general solution of (ii) as given

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in Art. 4 to the steady-state solution of (i) as given by equation (iii) to get the general solution of (i). Therefore

$$q = Q \sin(\omega t - \theta') + A_1\epsilon^{a_1 t} + A_2\epsilon^{a_2 t} \quad \ldots\ldots \text{(vi)}$$

is the general solution of (i) when equation (iii) of Art. 4 has unequal roots, and

$$q = Q \sin(\omega t - \theta') + B_1\epsilon^{at} + B_2 t\epsilon^{at} \quad \ldots\ldots \text{(vii)}$$

is the general solution of (i) when equation (iii) of Art. 4 has equal roots.

The second and third terms in the second members of (vi) and (vii) decay rapidly because a is always negative and a_1 and a_2 are negative when they are real, and their real parts are negative when they are complex.

To understand equations (vi) and (vii) let us consider the following example. A circuit containing L, R and C in series is connected to a.c. supply mains by closing a switch. At the instant of closing the switch the charge q on the condenser is zero and the current in the circuit $\left(\frac{dq}{dt}\right)$ is zero. Now according to (vi) and (vii) we are to think of the steady-state solution as existing from the beginning (when the switch is closed), but the steady state as given by equation (iii) cannot satisfy both of the conditions $q = 0$ and $\frac{dq}{dt} = 0$ at the instant of closing of the switch, and in general it satisfies neither of these conditions. But a transient state is added to the steady state in equations (vi) and (vii), the steady state and the transient state together satisfy the initial conditions, and the transient state dies out and leaves the steady state alone.

7. Some simple examples of steady-state and transient solutions. *Example* 1. A constant voltage E is suddenly applied to a circuit containing resistance R and inductance L in series, and it is desired to establish an expression for the growing value of the current i. Placing $C = \infty$ and writing i for dq/dt, equation (vii) of Art. 3 reduces to

$$L\frac{di}{dt} + Ri = E \quad \ldots\ldots\ldots\ldots\ldots \text{(i)}$$

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and it is convenient for easy reference to write down also the equation

$$L\frac{di}{dt} + Ri = 0 \quad \ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

The general solution of (ii) may be easily found by the method of Art. 4 of Chapter I, which gives

$$i = A\epsilon^{-\frac{R}{L}t} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(iii)}$$

where A is an undetermined constant of integration.

The steady-state solution of (i) is, of course, $i = E/R$, and therefore the general solution of (i) must be

$$i = \frac{E}{R} + A\epsilon^{-\frac{R}{L}t} \quad \ldots\ldots\ldots\ldots\ldots \text{(iv)}$$

But, necessarily, $i = 0$ when $t = 0$ (the instant of closing the switch), and we therefore find

$$A = -E/R$$

so that the final actual expression for the growing current i is

$$i = \frac{E}{R}\left(1 - \epsilon^{-\frac{R}{L}t}\right) \quad \ldots\ldots\ldots\ldots\ldots \text{(v)}$$

Example 2. A closed circuit containing resistance R and inductance L in series has in it a current I_0 at the instant $t = 0$, and it is desired to establish an expression for the decaying value of the current. In this case, no voltage impressed on the circuit, equation (vii) of Art. 3 reduces to

$$L\frac{di}{dt} + Ri = 0$$

whose general solution is given by equation (iii) above. But $i = I_0$ when $t = 0$ so that we get $A = I_0$ and equation (iii) becomes

$$i = I_0\epsilon^{-\frac{R}{L}t} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(vi)}$$

The steady state of the circuit in this example is, of course, $i = 0$.

Example 3. A constant voltage E is suddenly applied to a circuit containing resistance R and capacity C in series and it is desired to establish an expression for the growing charge

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q of the condenser. Writing $L = 0$ in equation (vii) of Art. 3 we get

$$R\frac{dq}{dt} + \frac{1}{C}\cdot q = E \quad \text{.....(vii)}$$

and it is convenient for easy reference to write down also the equation

$$R\frac{dq}{dt} + \frac{1}{C}\cdot q = 0 \quad \text{.....(viii)}$$

The general solution of (viii) is easily found to be

$$q = A\epsilon^{-\frac{t}{RC}} \quad \text{.....(ix)}$$

and the steady-state solution of (vii) is, of course, $q = CE$. Therefore the general solution of (vii) is

$$q = CE + A\epsilon^{-\frac{t}{RC}} \quad \text{.....(x)}$$

but $q = 0$ when $t = 0$ (the instant of closing the switch) so that we get $A = -CE$ and therefore equation (x) becomes

$$q = CE\left(1 - \epsilon^{-\frac{t}{RC}}\right) \quad \text{.....(xi)}$$

If we divide q by C we get an expression for the increasing voltage across the charging condenser, and if we differentiate and get an expression for dq/dt we will have an expression for the decreasing current which flows into the charging condenser.

Example 4. A condenser with an initial charge Q is connected to and allowed to discharge through a non-inductive resistance R, and it is desired to establish an expression for the decreasing charge q of the condenser.

In this case equation (vii) of Art. 3 reduces to

$$R\frac{dq}{dt} + \frac{1}{C}\cdot q = 0$$

whose general solution is given by equation (ix). But $q = Q$ at $t = 0$ so that $A = Q$ and equation (ix) becomes

$$q = Q\epsilon^{-\frac{t}{RC}} \quad \text{.....(xii)}$$

Example 5. A circuit containing resistance R and inductance L in series is suddenly connected, at time t', to a.c.

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supply mains between which the voltage is $e = \mathbf{E} \sin \omega t$, and it is desired to find an expression for the actual current i in the circuit.

In this case equation (vii) of Art. 3 reduces to

$$L\frac{di}{dt} + Ri = \mathbf{E} \sin \omega t \quad \text{.....(xiii)}$$

The steady-state solution of this equation is familiar to every student of electrical engineering and it is

$$i = \mathbf{I} \sin (\omega t - \theta) \quad \text{.....(xiv)}$$

where $$\mathbf{I} = \mathbf{E}/\sqrt{R^2 + \omega^2 L^2} \quad \text{.....(xv)}$$

and $$\tan \theta = \omega L/R \quad \text{.....(xvi)}$$

If we add the general solution of equation (ii) of this article to equation (xiv) we get the general solution of equation (xiii), namely

$$i = \mathbf{I} \sin (\omega t - \theta) + A\epsilon^{-\frac{R}{L}t} \quad \text{.....(xvii)}$$

But $i = 0$ when $t = 0$ so that from (xvii) we get

$$0 = \mathbf{I} \sin (-\theta) + A \quad \text{.....(xviii)}$$

Let us represent the value of $\mathbf{I} \sin (-\theta)$ by i_0 for brevity. Then i_0 is the value of the steady-state current at $t = 0$, and from (xviii) we have $A = -i_0$ so that equation (xvii) becomes

$$i = \mathbf{I} \sin (\omega t - \theta) - i_0\epsilon^{-\frac{R}{L}t} \quad \text{.....(xix)}$$

This equation is exhibited graphically in Fig. 9. The curve *ee* is a small portion of the voltage curve; the dotted curve i_s shows the steady-state current; the dot-dash curve i_d shows the decaying current $-i_0\epsilon^{-\frac{R}{L}t}$; and the curve *ii* shows the actual current which is at each instant equal to $i_s + i_d$. At the instant t' when the switch is closed the value of the steady-state current is i_0 and the value A of the decaying current i_d at this instant is equal to $-i_0$ so that the actual current at t' is zero.

If the switch is closed at an instant at which the steady-state current would be zero if it existed, then $i_0 = 0$ and

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the actual steady-state current starts out at once without any complications.

If the switch is closed at an instant at which the steady-state current would *not* be zero if it existed, then, as shown in Fig. 9, the steady-state current does exist from the start but the decaying current i_d makes the actual current zero at the start (when the switch is closed).

If the switch is closed at the instant at which the steady-state current would be at its maximum value **I**, if it existed,

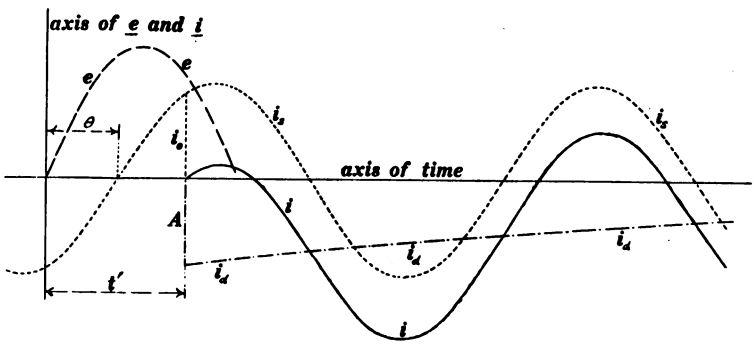

Fig. 9.

then $A = i_0 = \mathbf{I}$, and, if the transient term decays slowly, the actual current half-a-cycle after the closing of the switch will be nearly 2**I**.

Remark. The mathematical formulation of transients in complicated networks containing only resistances and lumped inductances and capacities is developed on the basis of the simple solutions given in this Appendix by making use of Kirchhoff's laws. These formulations are not used in this text and it is therefore not worth while to develope them here.

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APPENDIX D

USE OF CHARTS OF HYPERBOLIC FUNCTIONS

1. Definition of hyperbolic sine and hyperbolic cosine. The two expressions

$$\sinh w = \frac{\epsilon^{w} - \epsilon^{-w}}{2} \quad \ldots\ldots\ldots\ldots\text{(i)}$$

and

$$\cosh w = \frac{\epsilon^{w} + \epsilon^{-w}}{2} \quad \ldots\ldots\ldots\ldots\text{(ii)}$$

are the definitions of the hyperbolic sine of w and of the hyperbolic cosine of w. Exponential expressions for the ordinary circular sine and cosine are

$$\sin\ \theta = \frac{\epsilon^{j\theta} - \epsilon^{-j\theta}}{2} \quad \ldots\ldots\ldots\ldots\text{(iii)}$$

and

$$\cos\theta = \frac{\epsilon^{j\theta} + \epsilon^{-j\theta}}{2} \quad \ldots\ldots\ldots\ldots\text{(iv)}$$

and De Moivre's theorem is

$$\epsilon^{j\theta} = \cos\theta + j\sin\theta \quad \ldots\ldots\ldots\ldots\text{(v)}$$

(See Art. 6 of Appendix B.)

Exercises. (*a*) Using equation (v) prove equations (iii) and (iv).

(*b*) Show that $\sinh(-w) = -\sinh w$.

(*c*) Show that $\cosh(-w) = \cosh w$.

(*d*) Show that $\sinh[x + j(y \pm 2\pi n)] = \sinh(x + jy)$ if n is an integer.

(*e*) Show that $\cosh[x + j(y \pm 2\pi n)] = \cosh(x + jy)$ if n is an integer.

Expression of a circular angle in radians or in quadrants or in degrees. The quantity θ in equations (iii), (iv) and (v) is any pure number or numeric, but from geometrical considerations we are led to think of it as a *circular angle* expressed in

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radians. The value of θ can, however, be reduced to *quadrants* by dividing it by $\pi/2$ or to *degrees* by dividing it by $\pi/180$.

Similarly w in equations (i) and (ii) is any pure number or numeric, but from geometrical considerations we are led to think of it as an *hyperbolic angle* expressed in *radians*, and it can be reduced to *quadrants* by dividing it by $\pi/2$. The geometrical idea of one radian of hyperbolic angle is very different from the geometrical idea of one radian of circular angle, but from the algebraic point of view the radian is merely unity in either case.

Circular angle from an algebraic point of view may be complex (partly real and partly imaginary), and the same is true of hyperbolic angle. Complex hyperbolic angles are used in the following discussion.

Remark. Only the algebraic properties of the hyperbolic sine and cosine concern us when we wish to use these functions in numerical calculations. Therefore no attempt is here made to develope the geometrical properties of these functions.

2. Use of Kennelly Charts I and II. Charts I_A, I_B and I_C constitute a single extended chart for $\sinh \underline{\rho/\delta} = \underline{r/\gamma}$, and charts II_A and II_B constitute a single extended chart for $\cosh \underline{\rho/\delta} = \underline{r/\gamma}$. In both sets of charts the values of ρ and δ are affixed to the curves and the values of r and γ are the rectangular co-ordinates. The values of δ and γ are given in degrees.

Example. Given $\rho = 0{\cdot}90$ and $\delta = 59°$ to find $\sinh \underline{\rho/\delta}$ and $\cosh \underline{\rho/\delta}$. From chart I we find that $\sinh \underline{0{\cdot}90/59°} = \underline{0{\cdot}846/66°}$, and from chart II we find that $\cosh \underline{0{\cdot}90/59°} = \underline{0{\cdot}863/22{\cdot}83°}$.

There is no need to explain interpolation between curves, and the inverse use of any chart is evident when its direct use is understood.

3. Use of Kennelly Chart III. The three charts (one chart on sheet III_A and two charts on sheet III_B) are parts of a single chart to different scales, and this chart is for $\tanh \underline{\rho/\delta} =$

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$r\underline{/\gamma}$. The values of ρ and δ are affixed to the curves and the values of r and γ are the rectangular co-ordinates.

Examples.

$$\tanh 1{\cdot}7\underline{/77^\circ} = 2{\cdot}669\underline{/-11{\cdot}5^\circ}$$
$$\tanh 0{\cdot}6\underline{/45^\circ} = 0{\cdot}595\underline{/38{\cdot}2^\circ}$$
$$\tanh 1{\cdot}7\underline{/88^\circ} = 7{\cdot}04\underline{/-64{\cdot}9^\circ}$$

These values are taken, respectively, from the 1st, 2nd and 3rd charts for reasons which will be evident on inspection. There is no need to explain the inverse use of these charts.

4. Use of Kennelly Charts VII and VIII. These constitute one chart drawn to three different scales on sheets $VII\text{–}VIII_A$, $VII\text{–}VIII_B$ and $VII\text{–}VIII_C$; and this one chart is called chart VII when used for $\sinh(x + jq) = u + jv$, and it is called chart VIII when used for $\cosh(x + jq) = u + jv$.

The use of this chart, and of every other chart in which the "hyperbolic angle" appears in rectangular form, involves the bringing of any given complex quantity $a + jb$ (where a and b are pure numbers of any value and sign) **to a standard form $x + jq$ where x is positive and where q is expressed in quadrants and is positive and less than 4.**

Note that $\sinh(a + jb)$ is *reversed in sign* if $-(a + jb)$ is substituted for $+(a + jb)$, and note that $\cosh(a + jb)$ is *unchanged* if $-(a + jb)$ is substituted for $+(a + jb)$. Therefore :—

1. If a is positive take $x = a$.
2. If a is negative, reverse the signs of both a and b, that is to say take $x = -a$.
3. If b in case 1 (or after reversal of sign in case 2) is negative add to it the least multiple of 2π which will make it positive; or if b in either case is positive and greater than 2π, subtract from it a multiple of 2π which will leave it positive and less than 2π. Then divide the modified value of b by $\pi/2$ to reduce it to quadrants $= q$.

Note. The basis of rule 3 is that both $\sinh(a + jb)$ and

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cosh $(a + jb)$ are unchanged in value if any multiple of 2π is added to or subtracted from b, that is to say, sinh $(a + jb)$ and cosh $(a + jb)$ are periodic and their period is $j \times 2\pi$ just as the ordinary sine and cosine are periodic with a period equal to 2π.

4. If sinh $(a + jb)$ is desired, look up sinh $(x + jq) = u + jv$. This will be the correct value of sinh $(a + jb)$ if the original value of a was positive; but $-u - jv$ will be the correct value of sinh $(a + jb)$ if the original value of a was negative.
5. If cosh $(a + jb)$ is desired, look up cosh $(x + jq) = u + jv$. This will be the correct value of cosh $(a + jb)$ whatever the original sign of a.

To find sinh $(x + jq) = u + jv$. Place the chart with the long axes of the ellipses vertical. Then vertical ordinates show values of v, positive upwards and negative downwards; and abscissas show values of u, positive towards the right and negative towards the left. The numbers affixed to the ellipses are the values of x, and the numbers affixed to the hyperbolas are the *decimal parts* of the values of q in quadrants. Look at sheet VII–VIIIB (or at Fig. 1). The values of q begin at zero at S, reach 1 at C, reach 2 at S', reach 3 at C', and reach 4 at S.

Examples.

$$\sinh(1{\cdot}3 + j0{\cdot}5) = 1{\cdot}2 + j1{\cdot}39$$
$$\sinh[1{\cdot}3 + j(0{\cdot}5 \pm 2)] = -(1{\cdot}2 + j1{\cdot}39)$$
$$\sinh(1{\cdot}3 + j1{\cdot}5) = -1{\cdot}2 + j1{\cdot}39$$
$$\sinh(-1{\cdot}3 + j0{\cdot}5) = -1{\cdot}2 + j1{\cdot}39$$

To find cosh $(x + jq) = u + jv$. Place the chart with the long axes of the ellipses horizontal. Then vertical ordinates show values of v, positive upwards and negative downwards; and abscissas show values of u, positive towards the right and negative towards the left. The numbers affixed to the ellipses are the values of x, and the numbers affixed to the hyperbolas are the *decimal parts* of the values of q in quadrants. Look at sheet VII–VIIIB (or at Fig. 1). The values

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of q begin at zero at C, reach 1 at S', reach 2 at C', reach 3 at S, and reach 4 at C.

Examples.

$$\cosh(1{\cdot}3 + j0{\cdot}5) = 1{\cdot}39 + j1{\cdot}2$$
$$\cosh[1{\cdot}3 + j(0{\cdot}5 \pm 2)] = -(1{\cdot}39 + j1{\cdot}2)$$
$$\cosh(1{\cdot}3 + j1{\cdot}5) = -1{\cdot}39 + j1{\cdot}2$$
$$\cosh(-1{\cdot}3 + j0{\cdot}5) = 1{\cdot}39 - j1{\cdot}2$$

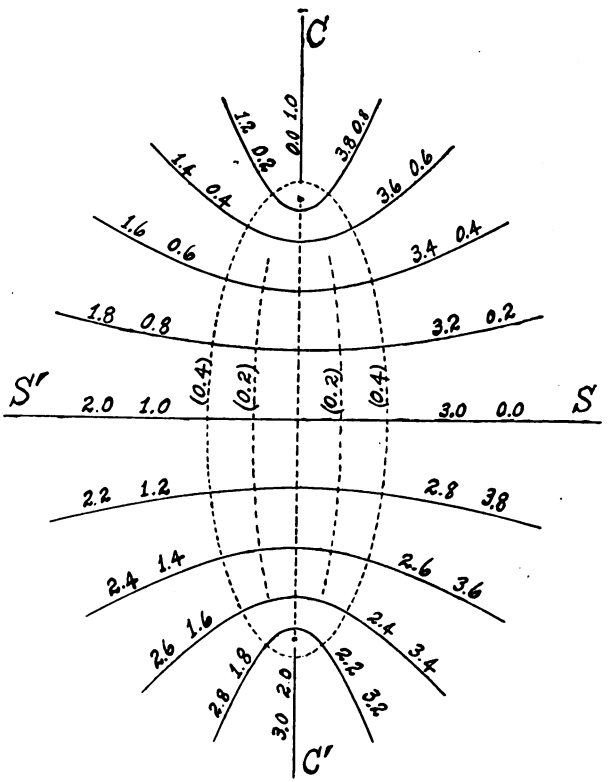

Fig. 1.

5. Use of Kennelly Chart IX for tanh $(x + jq) = u + jv$.

The hyperbolic tangent is defined by the equation

$$\tanh w = \sinh w/\cosh w = \frac{\epsilon^{2w} - 1}{\epsilon^{-2w} + 1}$$

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and $\tanh w$ is periodic, its period being $j\pi$, so that if q, as given, is greater than 2, subtract 2 from it and use the remainder for q.

If the real part of the original given hyperbolic angle $a + jb$ was negative (before being reduced to $x + jq$ where x is positive and q is in quadrants), reverse the sign of $\tanh(x+jq)$ to get the correct value of $\tanh(a + jb)$.

The values of x are affixed to the "encircling" circles in chart IX, the values of q are affixed to the "intersecting" circles, the values of u are abscissas, and the values of v are ordinates.

Examples. $\tanh(0{\cdot}5 + j1{\cdot}5) = 0{\cdot}75 - j0{\cdot}65$
$\tanh(0{\cdot}5 - j1{\cdot}5) = 0{\cdot}75 + j0{\cdot}65$
$\tanh(-0{\cdot}5 + j1{\cdot}5) = -0{\cdot}75 + j0{\cdot}65$
$\tanh -(0{\cdot}5 + j1{\cdot}5) = -(0{\cdot}75 - 0{\cdot}65)$

6. Use of Kennelly Chart X–XI for sinh $(x + jq) = r\underline{/\gamma}$ **or for cosh** $(x + jq) = r\underline{/\gamma}$. Note that chart XI is an extension of chart X towards the right.

The values of x are affixed to the curves which are nearly vertical in chart XI and become strongly curved about the focal point in chart X.

The *decimal parts* of q are affixed to the curves which are nearly horizontal in XI and oval shaped in X.

For $\sinh(x + jq)$ the values of q start at zero at the bottom of the chart and run up to 2 at the 180°–90° line.

For $\cosh(x + jq)$ the values of q start at zero at the middle horizontal line CC, run up to 1 at the top (180°–90° line), and then start at 1 at the bottom of the chart and run up to 2 at the middle line CC.

To find sinh $(x + jq) = r\underline{/\gamma}$ enter the chart at the lower left-hand corner and go to the right until a nearly vertical x-curve is reached with the given value of x, then follow this x-curve upwards until the given value of q is found on a crossing q-curve. Read r on abscissa scale at top or bottom, and read γ on ordinate scale on left.

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To find cosh $(x + jq) = r\underline{/\gamma}$ enter the chart at the center of the left-hand side at the point C and go to the right (past the focal point) until an x-curve is found for the given value of x, then follow this x-curve upwards until the given value of q is found on a crossing q-curve. Read r as before, and read γ on ordinate scale on the right.

Remark. It is easily shown that if 2 quadrants are subtracted from q 180° will be subtracted from γ for both sinh $(x + jq)$ and cosh $(x + jq)$. Hence if the given value of q is greater than 2, subtract 2 from it and look up r and γ for the given value of x and this reduced value of q, and add 180° to the value of γ so found.

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APPENDIX E

THE CORONA

1. Air as an electrical conductor. When a charged gold-leaf electroscope is left standing it soon loses its charge and the leaves fall together. This loss of charge is usually due almost wholly to the leakage flow of electricity along the imperfectly insulating supports of the gold leaves, but it has long been known that it is due in part to the leakage flow of electricity through the surrounding air. That is to say, air ordinarily conducts electricity to a slight extent.

There are a number of influences which cause air (or any gas) to become a fairly good electrical conductor. Thus gas which is drawn from a flame or from the neighborhood of glowing metal or carbon is a fairly good conductor; gas which has been drawn from a region through which an electric discharge has recently passed is a fairly good conductor; a gas becomes a fairly good conductor under the action of X-rays or under the action of the radiations from radio-active substances.

The conductivity which is imparted to a gas by these various agencies gradually disappears when the conducting gas is left standing, and the conductivity is quickly destroyed by filtering the gas through glass wool or by placing the gas for a few moments between electrically charged metal plates. This effect of filtration seems to show that the conductivity of the gas is due to something which is mixed with the gas, and the effect of the electric field (between the two charged plates) seems to show that this something is charged with electricity and is dragged out of the gas by the electric field. From such considerations as these the hypothesis was originated that the electrical conductivity of a gas is due to *electrically*

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charged particles floating around in the gas. This hypothesis has been used extensively and with remarkable success in the study of electrical discharge through gases and in the study of radio-activity, and the hypothesis together with the associated group of clearly defined fundamental ideas is now called the *electron theory*.

2. The electron theory. The most important of the fundamental ideas of the electron theory are the now widely accepted ideas of *the electron*, *the proton*, the Rutherford-Bohr conception of *the atom*, and the conception of *the ion*.

The electron is the "atom" of negative electricity. Its value e is $-1{\cdot}59 \times 10^{-21}$ abcoulombs or $-4{\cdot}77 \times 10^{-10}$ stat-coulombs, its mass is 1/1835 of the mass of the hydrogen atom, and its radius is of the order of 2×10^{-13} centimeter or about one fifty-thousandth of the radius of a hydrogen atom which is about a hundred-millionth of a centimeter. It is, of course, absurd to speak of a mere "idea" in this concrete way, but the electron is now known to exist objectively.* The cathode rays are electrons which are thrown off at high velocities from the cathode of a Crookes tube. The β-rays from the radio-active elements are electrons which are expelled at extremely high velocities from the disintegrating atoms of these elements.

The proton is the "atom" of positive electricity. Its value is the same as the electron, but, of course, positive. It has never been recognizably isolated, that is to say, the proton is still hypothetical.

The atom of any chemical element, according to the Rutherford-Bohr conception, is an exceedingly minute positively charged nucleus with electrons revolving around it. The positive nucleus is supposed to be built up of closely packed electrons and protons with an excess of N protons so that the net nuclear charge is $+Ne$. In the neutral state

* Some of the most important of the experimental evidences of the electron are brought together by Professor R. A. Millikan in a book entitled *The Electron*, Chicago University Press, 1917.

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there are N electrons revolving around the nucleus so that the net charge of the neutral atom is zero. The number N is called the atomic number. Thus the atomic number of hydrogen is 1, the atomic number of helium is 2, the atomic number of lithium is 3, the atomic number of sodium is 11, and so on.

A simple ion is an atom from which a negatively charged electron has been detached thus leaving the remainder of the atom (the simple ion) positively charged; or a simple ion may be a neutral atom to which an extra electron has become attached thus giving the atom (the simple ion) a negative charge.

The compound ion is an aggregate of two or more atoms or molecules with an extra electron attached to it thus giving it a negative charge, or from which a negatively charged electron has been detached thus leaving the aggregate positively charged. In highly rarefied gases electrons and simple ions, only, exist; whereas compound ions may exist in gases at moderate or high pressure.

When a gas contains free electrons and ions the gas is said to be *ionized*. An ionized gas is an electrical conductor. When an ionized gas is left standing the free negative electrons and positive ions slowly find each other and combine to form neutral atoms and the gas loses its conducting power. When an ionized gas is passed slowly between oppositely charged metal plates the negative electrons are drawn to the positively charged plate and the positive ions are drawn to the negatively charged plate and the gas loses its conducting power. When an ionized gas is passed through glass wool the charged particles (electrons and positive ions) are sure to come very near to the fibers of the glass wool, and when a charged particle thus comes very near to a fiber it induces an opposite charge on the fiber, is then attracted by the fiber, and is left clinging to the fiber.*

* This statement ignores the effect of the high inductivity of the fiber material which, independently of any induced charges in the ordinary sense, causes the glass fibers to attract and hold negative electrons or positive or negative ions.

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Normal states and excited states. An atom has a *normal state* when the electrons are in the smallest possible orbits, and an atom has a series of *excited states* when one or more of the electrons have been " lifted " to larger orbits. An excited atom has more energy than a normal atom, and when an excited atom changes to a lower state of excitation (a lower energy state) or to the normal state energy is emitted as radiation.

When an electron is entirely removed from an atom the atom is said to be ionized (*singly ionized*), when two electrons are entirely removed the atom is said to be *doubly ionized*, etc.

Ionized states are extreme examples of excited states. An atom may be excited or ionized by the absorbtion of radiation (of the proper frequency) or by impact with other atoms (?) or electrons or ions ; and a highly excited state or complete ionization is produced in stages, a comparatively small amount of energy being required for each stage.

A molecule also has a normal state and a series of excited states, and nearly everything stated above as to the excited (or ionized) state of an atom applies also to the excited state of a molecule.

When the light from a gas through which an electric discharge is passing is analysed in a spectroscope one sees a large number of very sharp spectrum lines, and a part of the spectrum, usually a fairly narrow part, is continuous. The light which gives the sharp lines is known to come from the " dropping " of molecules or atoms from excited states of high energy to excited states of lower energy or to the normal state, whereas the light which forms the continuous spectrum is known to come from the recombination of free electrons and free positive ions.

No careful energy measurements have been made, but a glance at the spectrum of a gas through which an electric discharge is passing convinces one that a vastly greater amount of light comes from the " dropping " of excited states than comes from the recombination of free electrons and positive ions so that the collisions of the moving electrons

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and ions with the gas molecules (see Art. 4) must produce a vastly greater number of excited molecules and atoms than of free electrons and ions; although a greater number of the latter may accumulate because the "life" of an excited molecule or atom is usually very short.

When the energy of the electric discharge through a gas is very concentrated, as in the electric spark, the spectroscope gives unmistakable evidence that a simple positive ion can be brought into an excited state or even ionized (doubly ionized).

3. Mean free paths. According to the kinetic theory of gases a molecule of a gas travels, on the average, a certain distance l between successive collisions with other molecules of the gas, and this distance l is called the *mean free path* of the molecule. Thus the molecules of O and N in the air at 25° C. and at 760 mm. pressure have a mean free path of about $9{\cdot}7 \times 10^{-6}$ centimeter, and the mean free path is doubled if the density is halved by a reduction of pressure. Rise of temperature (at constant pressure) increases the mean free path chiefly because of the decrease of density and only to a very slight extent because of the increase of molecular velocities due to the rise of temperature.

The mean free path of a positive ion in the air (a simple ion) is somewhat greater than the mean free paths of the O and N molecules because the simple ion is smaller than the molecules; and the mean free path of an electron in the air is $4\sqrt{2}$ times the mean free paths of the O and N molecules. The factor 4 comes from the extremely small size of the electron as compared with the molecule, and the factor $\sqrt{2}$ comes from the very great to-and-fro velocity of a free electron as compared with the to-and-fro velocities of the air molecules.

4. Ionization by collision. Consider a body of gas between oppositely charged flat metal plates. The electric field between the plates is nearly uniform, this field exerts a pull on any electron (or ion) which may be floating around in the gas, and a certain amount of velocity is imparted to the electron (or

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ion) between successive collisions of the electron (or ion) with the molecules of the gas. If this imparted velocity is sufficiently great the electron (or ion) breaks the molecules of the gas to pieces or ionizes them when it collides with them, and when a molecule of a gas is thus broken to pieces a new electron and a new ion are produced.* This process is called *ionization by collision.*

If the velocity of the electron (or ion) when it collides with a gas molecule is too small to produce complete ionization it may produce an excited state (see Art. 2).

The mean free path of the electron (or ion) is the average distance traveled between collisions, and a definite fraction of all actual paths exceed the mean by a certain amount. This fraction can be estimated by a developement of the kinetic theory, but it is sufficient here to state that a certain fraction of the actual paths would be 1·2 times the mean, a smaller fraction would be 1·4 times the mean, a still smaller fraction would be 1·6 times the mean, etc.

In a very weak electric field an occasional electron (or ion) might travel far enough between collisons to enable it to ionize an air molecule, but this result would be extremely rare. When the field intensity reaches about 13,000 volts per centimeter a barely perceptible fraction† of the free paths of the electrons are long enough to enable the electrons to acquire

* We are not here concerned with the fact that a single collision may produce more than one new electron and one new ion.

† Given a body of air between metal plates *with a constant outside agency which is producing electrons in the gas, X-rays for example.* When the voltage between the plates is increased, soon all of the electrons (and ions) as they are formed are drawn to the plates before they find each other and re-combine, and the current from plate to plate then has reached what is called the *saturation value.* No more current can flow unless more electrons and ions are liberated. When the electric field intensity between the plates reaches about 13,000 volts per centimeter there begins to be an appreciable current in excess of the saturation current, which shows that the electrons as they are pulled along by a field of 13,000 volts per cm. produce an appreciable amount of ionization by collision. Perhaps in a field of 13,000 volts per cm. much of the ionization by collision may be step by step excitation leading to final complete ionization; however this may be, it is certain that ionization by collision begins to be appreciable in a field of 13,000 volts per cm. in air at 25° C. and 760 mm.

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sufficient velocity to ionize the air molecules (let it be remembered that we are considering air at 25° C. and at 760 mm. pressure).

5. The electric spark in a gas. Let us consider the two oppositely charged metal plates A and B in Fig. 1, and let us assume that the air between A and B is at 25° C. and at 760 mm. pressure. The electric field between A and B is approximately uniform.

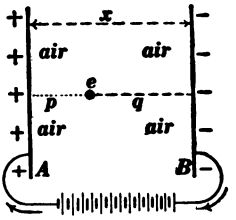

Fig. 1.

The body of air between A and B always contains a few stray electrons and positive ions,* and as soon as the battery is connected between the plates, as shown, the electric field which is set up between A and B exerts a force on each electron (positive ions need not be considered). When this electric field is increased to about 13,000 volts per centimeter a very small (but appreciable) fraction of the free-path movements of the electrons are long enough for the electrons to gain sufficient velocity between collisions with air molecules to enable them (the electrons) to ionize by collision, and when the electric field is increased to about 30,000 volts per cm. a much larger fraction of the free-path movements of the electrons are long enough to lead to ionization by collision.

The electron avalanche. Let us consider a body of air at 25° C. and 760 mm. pressure between parallel flat metal plates 5 cm. apart. With a given voltage between the plates let us assume that $1/m$th of the free paths described by the electrons lead to ionization by collision. Then, given N electrons to start with, N/m new electrons will be formed after one free-path movement of all, and the number of electrons will then be $N\left(1 + \frac{1}{m}\right)$, after the next free-path movement the

* Produced by the ionizing action of light or by the ionizing action of weak X-rays which exist everywhere, or by the minute amount of radioactive substance which exists everywhere.

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number will be $N\left(1 + \frac{1}{m}\right)^2$, and after n free-path movements the number of electrons will be $N\left(1 + \frac{1}{m}\right)^n$ if the moving group of electrons does not begin to exhaust the supply of air molecules, and if recombination of positive ions and electrons is negligible.

The 5 cm. distance between the plates amounts to about 100,000 mean free paths, so that for an electron starting near the negatively charged plate and traveling to the positively charged plate we would have $n = 100{,}000$. Therefore one electron starting near the negatively charged plate would become $\left(1 + \frac{1}{m}\right)^{100{,}000}$ electrons by the time it has reached the positvely charged plate. The values of $\left(1 + \frac{1}{m}\right)^{100{,}000}$ for several values of m are exhibited in the following table I. Let us also consider the values of $\left(1 + \frac{1}{m}\right)^n$ for a chosen value of m and for various distances (various values of n) as shown in table II.

TABLE I FOR $n = 100{,}000$

Value of m	Value of $\left(1 + \frac{1}{m}\right)^n$
1,000	10^{43} an unthinkably large number
5,000	$10^{8\cdot6}$ a very large number
10,000	20,000
50,000	7·3
100,000	2·7
1,000,000 etc.	1·1 etc.

TABLE II FOR $m = 1000$

Distance	Value of n	Value of $\left(1 + \frac{1}{m}\right)^n$
5·00 cm.	100,000	10^{43} an unthinkably large number
1·00 ,,	20,000	4×10^8 a very large number
0·50 ,,	10,000	20,000
0·10 ,,	2,000	7·3
0·05 ,,	1,000	2·7
0·02 ,,	400	1·5
0·01 ,, etc.	200 etc.	1·2 etc.

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Tables I and II show that under proper conditions a single original electron becomes an enormously large number after traveling a moderate distance. This is, of course, largely a mental picture, but something of this kind does actually occur under proper conditions, and this kind of thing is called an *electron avalanche.*

Table I shows that the number of electrons produced after 5 cm. of travel changes from an insignificant number to a very large number when m changes from 50,000 to 5,000; and as a matter of fact this change of m occurs in a fairly narrow range of electric field intensity. Therefore the electron avalanche becomes marked at a fairly definite electric field intensity. At 13,000 volts per cm. the value of m is, perhaps, about one million, and the value $m = 5000$ occurs at about 30,000 volts per cm.

Table II shows that, for a given value of m, the developement of the electron avalanche requires a certain distance, this distance is about 1 cm. in this table.

Most of the electrons in an electron avalanche travel along in a comparatively small group, but a fraction of them become spread out along the path of the avalanche and the result is somewhat as follows:—

(*a*) A very considerable amount of heat is developed along the avalanche path, the air expands, the mean free paths are increased and ionization by collision is increased;

(*b*) A portion, at least, of the avalanche path becomes a fairly good electrical conductor and the electric field is strengthened very greatly in the portions of the avalanche path which are poorly conducting or in the space between the head of the avalanche and the negatively charged plate*;

(*c*) Among the enormous numbers of impacts of electrons

* Some of the electric flux lines that start from plate A in Fig. 1 terminate on electrons instead of going on to plate B and some of the electric flux lines which terminate on plate B do not come from plate A but from positive ions in the intervening space. Accordingly the presence of free ions and electrons between A and B (when no longer intimately intermingled) leads to a non-uniform field between A and B.

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with air molecules some of the air molecules and perhaps also atoms of oxygen and nitrogen become excited, or some of the positive ions may become excited ; and

(*d*) Due to rise of temperature, local intensification of field and presence of excited molecules, atoms and ions, the positive ions as they are pulled towards the negatively charged plate begin to produce ionization by collision so that an indefinite increase of ionization takes place all along the path of the avalanche from plate to plate and an electric spark is produced.

If the voltage is held just below the sparking value numerous electron avalanches occur throughout the body of air, and if a favorable combination of circumstances should come about by chance (local heating, local concentration of electric field, production of excited molecules, etc.) a spark may be formed slightly below the usual sparking voltage. The thing as a whole is extremely complicated (some complications have not even been hinted at, for example, minute irregularities of the surfaces of the plates cause local concentrations of the electric field), but it is certain that a spark, which is a narrow path of extremely intense ionization bridging from plate to plate, does not occur until positive ions begin to produce ionization by collision somewhere and presumably near the negatively charged plate.

6. The dielectric strength of air. The electric field intensity (voltage gradient) which is just sufficient to cause cumulative ionization as described in Art. 5, is called the *dielectric strength of air*. At a pressure of 76 cm. and at 25° C. the dielectric strength of air is very close to 30,000 volts per centimeter.

The dielectric strength of a gas (at given temperature) should be proportional to the pressure because halving the pressure would halve the density and double the mean free paths so that an electric field half as intense should be able to give sufficient velocity to the electrons and ions between collisions to enable them to ionize by collision. In fact the dielectric strength of air (at a given temperature) is proportional

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to the pressure, being about 3000 volts per cm. at 1/10 of an atmosphere, about 30,000 volts per cm. at 1 atmosphere, and about 300,000 volts per cm. at 10 atmospheres.

Taking 30,000 volts per cm. as the dielectric strength of air at 25° C. and at 76 cm. pressure and assuming that the dielectric strength is proportional to the density (and independent of temperature except in so far as a change of temperature may cause a change of density) it is easy to derive the following

$$s = 30{,}000 \times \frac{298p}{76(273 + t)} \quad \ldots\ldots\ldots\ldots \text{(i)}$$

where s is the dielectric strength in volts per cm. of air at a pressure of p cm. and at a temperature of t° C.

Remark. The voltage required to spark across an air gap g cm. long between parallel flat plates is sg volts if the air gap is fairly long, but if the air gap is short a much greater voltage than sg volts may be required to spark across it. This is because the avalanche effect which is described in Art. 5 cannot develope in a short stretch of air as may be understood by studying table II in Art. 5. For example, if the flat plates are only 0·0001 cm. apart a voltage gradient of many times 30,000 volts per cm. is required to produce a spark.

7. The corona discharge. The curved lines in Fig. 2 show, approximately, the mode of distribution of the electric field between two parallel wires; the plane of the paper is at right angles to the wires, and the small circles represent the wires in section. The electric field is much more intense near the wires where the lines of force are crowded together than it is at greater distances from the wires where the lines of force spread more widely apart. If the wires are very fine the electric field may be very intense in a very small region around each wire even when the voltage between the wires is not very great.

Imagine the electromotive force between a given pair of wires to be increased until the field intensity in a region near the wires is at least 30,000 volts per centimeter. Then a stray

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electron near the wire will start an ionizing action which will *continue* indefinitely in the immediate neighborhood of the wires, and fill this region with ions and electrons, constituting the corona discharge. The corona takes its name from the faint bluish light which is emitted from the region where ionization by collision takes place. This light is probably emitted by the recombination of electrons and ions, and by the " dropping " of molecules and atoms from excited states.

The ionization in the corona does not increase indefinitely

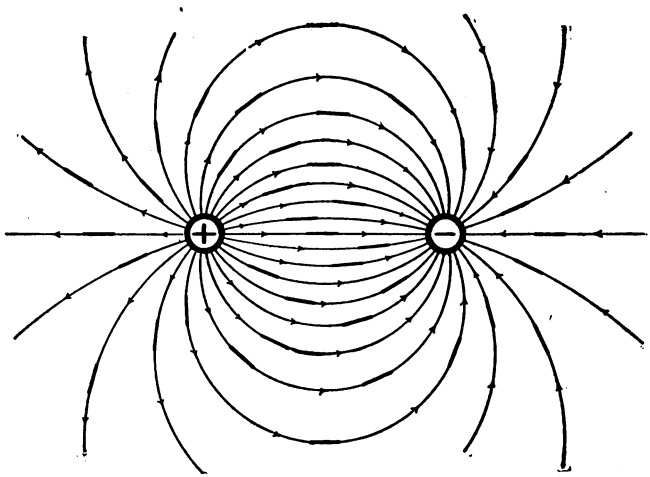

Fig. 2.

and form a spark. This is because the electric field is the most intense in the neighborhood of the wires, as shown in Fig. 2. Accordingly, when the region near the wires is just above the ionizing voltage gradient, the electric field in the space midway between wires is very weak, and will produce no ionization. The corona discharged is thus confined to a small region near the wires, where the field intensity is sufficiently great to cause ionization.

8. The Cottrell process for the precipitation of dust and smoke particles. A fine wire is stretched along the axis of a metal tube; a steady (*not* alternating) voltage high enough to

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produce corona discharge around the fine wire is connected between wire and tube ; and the result is that the particles of smoke or dust are deposited on the walls of the tube and to a very slight extent on the wire.

The most satisfactory arrangement for demonstrating the Cottrell process is shown in Fig. 3. A fine wire *AA* , supported by two insulating glass posts, is stretched through a large glass tube, and a piece of strap iron *BB* is laid along the bottom of the tube as shown in the figure. The fine wire and iron strap are connected to the terminals of a Toepler-Holtz influence machine, and the smoke to be cleared is blown in at one end of the tube. The effect is most strikingly shown by short-circuiting the influence machine by means of a small metal rod,

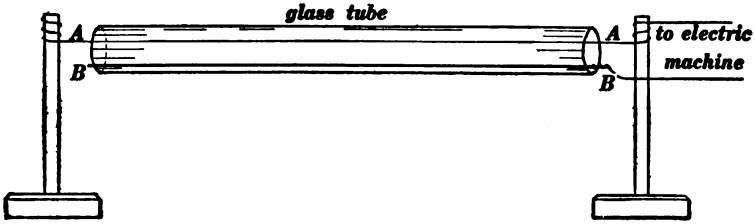

Fig. 3.

filling the tube with smoke, and then suddenly removing the short circuit.

The Cottrell separator works almost equally well whether the wire be made positive or negative. If the wire is positive, great numbers of ions with their positive charges travel out from the corona, these ions attach themselves to the particles of dust or smoke, these particles thus become positively charged, and the electric field between wire and tube drags the positively charged particles to the negatively charged wall of the tube. If the wire is negative, great numbers of electrons with their negative charges travel out from the corona and attach themselves to the smoke or dust particles, and the particles thus negatively charged are attracted by and travel to the positively charged wall of the tube.

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9. Derivation of expression for voltage between line wires for which corona begins. By using a slight modification of the method employed in Art. 2 of Appendix A it can be shown that

$$f = 0{\cdot}2172 \times \frac{E}{x \log_{10}\left(\frac{D}{R}\right)} \quad \ldots\ldots\ldots\ldots \text{(i)}$$

where f is the electric field intensity (in volts per cm. when x is in centimeters) at a point distant x from the axis of one of the line wires, E is the voltage between the line wires, D is the distance apart of the line wires (center to center), and R is the radius of the wires. This is an approximate formula and it is to be used only for values of x which do not greatly exceed R, but the value of x must not be less than R.

Experiments by Peek have shown that the electric field intensity f must be equal to or greater than s throughout a stretch from $x = R$ to $x = R + 0{\cdot}30\sqrt{R}$ (when R is expressed in centimeters) in order that corona may form. Therefore, keeping in mind that R must be expressed in centimeters, we may put $R + 0{\cdot}30\sqrt{R}$ for x and s for f in equation (i) and we get

$$E = s(R + 0{\cdot}30\sqrt{R}) \log_{10}\left(\frac{D}{R}\right) \Big/ 0{\cdot}2172 \quad \ldots\ldots \text{(ii)}$$

or, using the value of s from equation (i) of Art. 6, we get

$$E = \frac{30{,}000}{0{\cdot}2172} \times \frac{298p}{76(273 + t)} \times (R + 0{\cdot}30\sqrt{R}) \log_{10}\left(\frac{D}{R}\right) \quad \text{(iii)}$$

where E is the critical voltage for which corona begins. If it is desired to express R in inches then $30{,}000 \times 2{\cdot}54$ must be written in place of 30,000, and $R + 0{\cdot}188\sqrt{R}$ must be written for $R + 0{\cdot}30\sqrt{R}$ in equation (iii).

Remark. Throughout the above discussion steady (*not* alternating) voltages are understood, and equation (iii) gives the steady voltage required to produce corona; or, equation (iii) gives the maximum value of the alternating voltage required to produce corona.

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Example. What voltage between wires is required to produce corona when $R = 0{\cdot}5$ cm., $D = 45$ cm., $p = 76$ cm., and $t = 25°$ C? Answer 192,000 volts steady, or 135,800 effective alternating voltage if the alternating voltage is harmonic.

10. Power loss due to corona. The corona involves an appreciable flow of leakage current from wire to wire on a transmission line, and causes an appreciable loss of power. This loss of power on a long line becomes so great that it is always necessary to operate long lines at voltages below the corona-producing voltage.

It is evident from equation (iii) of the foregoing article that the corona-producing voltage increases with the size of the line wires so that if a small branch line leads off from a very high-voltage transmission line it often happens that a larger wire is demanded on this branch line for the elimination of corona than would be required for other reasons, to limit the voltage drop on the branch line, for example.

Equation (iii) of the foregoing article is important to the engineer who wishes to determine the size (and spacing) of line wires so that corona will not appear for any given voltage between wires.

11. Corona on stranded wires. Equations (i), (ii) and (iii) of Art. 9 of this Appendix apply to smooth cylindrical wires. When the line wires become roughened by weathering the critical corona-producing voltage is lowered because the roughened surface gives local points of high voltage gradient which causes local spots of corona when the voltage between wires is too low to give corona over the whole surfaces of the wires.

Also with stranded line wires corona appears at lower voltages than would be the case if the stranded wires or cables were replaced by cylindrical wires of the same outside diameter.

It is customary to introduce a multiplier m into the right-hand member of equation (ii) or equation (iii) of Art. 9 to

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allow for the effects of weathering of cylindrical wires, or to give the corona-producing voltage for stranded line wires (cables). In the latter case R is to be taken as the radius of the envelope of the stranded conductor or cable. The values of m are as follows :—

For smooth, polished cylindrical wires	$m = 1{\cdot}00$
For cylindrical wires, weathered	$m = 1{\cdot}00$ to $0{\cdot}90$
For cables (stranded conductors)	$m = 0{\cdot}85$ to $0{\cdot}70$

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APPENDIX F

INDUCTIVE INTERFERENCE BETWEEN A POWER LINE AND A TELEPHONE LINE

1. Disturbance in telephones due to atmospheric electricity or "static." In early days the telephone circuit consisted of a single line wire with a ground return, and a very troublesome noise was often produced in the telephones at both ends of a line as the accumulated electricity on the line flowed to ground through the telephones or as the induced currents in the line due to atmospheric disturbances flowed through the telephones. This old trouble has been almost entirely eliminated by the

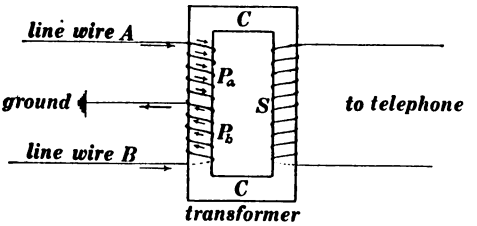

Fig. 1.

use of the two-wire telephone line with a telephone transformer arranged as in Fig. 1 at each end of the line. The central point in the primary coil of the transformer is grounded. Stray currents coming into a telephone over both line wires pass in *opposite* directions around the transformer core *CC* in the two halves P_a and P_b of the primary coil thus neutralizing each other in so far as the telephone is concerned, whereas the voice current comes in on one line wire and goes out on the other and flows in the *same direction* around the transformer core in the two halves of the primary coil, thus

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producing induced voice current in the secondary coil S and in the telephone.

The use of the two-wire telephone line and the use of the telephone transformer with the middle point of its primary coil grounded as shown in Fig. 1 are important helps in the elimination of telephone troubles due to the inductive action of a power line on a telephone line. It is only necessary to prevent the production of unequal currents in the two wires of the telephone line. This statement becomes more intelligible if it is stated that opposite currents must not be produced in the two wires of a telephone line.

2. Electromagnetic inductive interference. Opposite currents are induced in the two wires of a telephone line by the alternating magnetic flux (due to an adjoining power line) which passes between the telephone wires and by transposing the wires of the telephone line at equal distances these inductive effects can be eliminated.

If the power line were a single-phase line these inductive effects could be eliminated by transposing the wires of the power line at equal distances.

If the power line is a three-phase line *with neutral grounded at both ends* the 3rd, 9th, 15th, etc., harmonics of current flow outwards in all three line wires simultaneously and backwards in all three wires simulataneously as explained in Chapter VII, and the disturbing influence of these harmonics on an adjacent telephone line cannot be eliminated by transposing the wires of the power line although the disturbing influence can in this case be eliminated by transposing the wires of the telephone line.

By using the methods of Appendix A it is fairly easy to set up an equation for calculating the net alternating voltage in the telephone circuit in any case and by dividing this voltage by the impedance of the telephone circuit (neglecting the wire-to-wire capacity of the telephone line) we get a fairly accurate value of the alternating current induced in the telephone circuit.

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3. Electrostatic inductive interference. The alternating electric charges on the wires of a power line produce an alternating voltage e between the wires of an adjacent telephone line and the currents (alternating, of course) produced in the

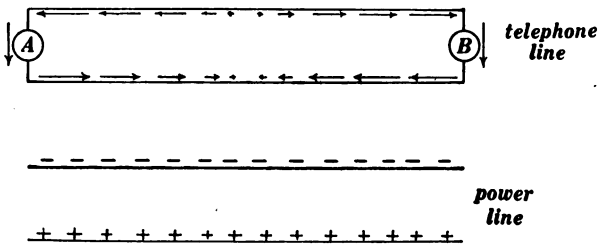

Fig. 2.

Arrows show currents induced while the charges, as indicated, are increasing.

telephone circuit by e are shown, for a given instant, by the arrows in Fig. 2. The alternating-current i in each telephone is given approximately by $i = e/\sqrt{R^2 + (\omega L - 1/\omega C)^2}$ where R is the resistance of about $\frac{1}{4}$ of the telephone line plus the resistance of the apparatus at A (or at B), L is the inductance

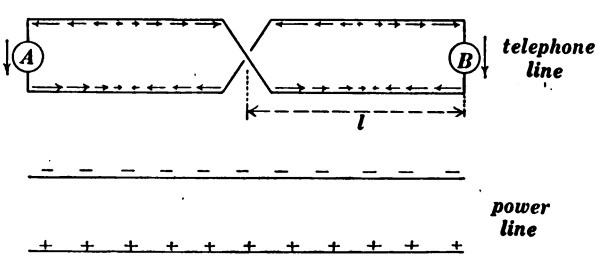

Fig. 3.

Arrows show currents induced while charges, as indicated, are increasing.

of about $\frac{1}{4}$ of the telephone line plus the inductance of the apparatus at A (or at B), and C is the wire-to-wire capacity of $\frac{1}{2}$ of the telephone line. The value of e can be calculated for any particular case by a slight modification of the method used in Appendix A.

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The alternating current through A (or through B) in Fig. 2 can be greatly reduced by transposing the wires of the telephone line or by transposing the wires of the power line as shown in Figs. 3 and 4, but in no case can the alternating current through A (or through B) be entirely eliminated in this way, as may be understood by a careful study of Figs. 3 and 4. If the alternating charges on the wires of the power

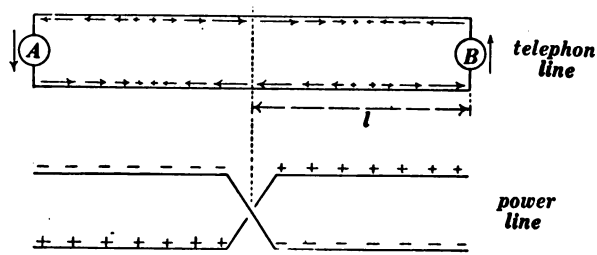

Fig. 4.

Arrows show currents induced while charges, as indicated, are increasing.

line do not add up to zero (because of unbalancing effects) the effect of transposing the wires of the power line is quite unsatisfactory.

A careful study of Fig. 3 (or of Fig. 4) will show that the alternating current arising in one-half of a transposition section flows through each telephone so that the transposition sections must be made very short if this alternating current is to be very small.

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APPENDIX G

HOW CAN AN ELEMENT OF A TWO-WIRE LINE BE CONSIDERED AS A CIRCUIT HAVING A DEFINITE INDUCTANCE ?

The question is often raised as to the legitimacy of the methods of Art. 7 Chapter I and Art. 4 Chapter III where an element of a two-wire line is treated as a complete circuit having a definite inductance.

One aspect of this question is very simple. Let us, for example, think of current as flowing only in one short element of a two-wire line. Then the increasing voltage across one end of the line element and the decreasing voltage across the other end of the line element constitute what Maxwell called displacement currents which complete the circuit of the current flowing in the line element.

There is, however, a much more complicated aspect of the question, a statement of which necessitates some preliminary remarks. Consider a *steady current* flowing in a *given circuit.* From the point of view of Maxwell's equations this current is a parameter whose value completely determines the intensity of the magnetic field everywhere. If the current i is doubled the field intensity is everywhere doubled, the magnetic energy is everywhere quadrupled and the total magnetic energy is quadrupled. That is to say, the total magnetic energy can be expressed in the form $\frac{1}{2}Li^2$ where L, the inductance, is a circuit constant. In general, however, if the current in the circuit is changing rapidly, the total energy of the associated magnetic field may not be exactly expressible as $\frac{1}{2}Li^2$ and the given circuit may not have a definite inductance. A similar statement applies to a pair of insulated conductors with voltage e between them ; when e is constant the total

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energy of the electric field is $\frac{1}{2}Ce^2$ but when e is changing rapidly the total energy of the electric field may not be expressible as $\frac{1}{2}Ce^2$. *Inductance and capacity are both steady-state conceptions.*

Given a long two-wire transmission line, and let x be a co-ordinate measured along the line. Consider a thin plane layer of the space surrounding the line, this layer being at right angles to the line and of thickness dx, and the abscissa of the thin plane layer being x measured from any fixed origin. The question now is as to whether the electromagnetic action throughout this plane layer can be expressed in terms of the current i in the two-wire line and the voltage e across the two wire-line at the point x *as if* e *and* i *were constant.* The answer is Yes, if the two-wire line is long and if the line wires have negligible resistance.*

The foregoing statement is somewhat vague, but the following statements cover the case precisely. First we will call to mind the well-known constant, "two-dimensional" distributions of electric and magnetic fields about a two-wire line, and then we will make use of these distributions to specify a new distribution which will turn out to be the desired solution of Maxwell's equations for a long two-wire line of negligible resistance.

(*a*) Consider the well-known magnetic field distribution about an infinitely long two-wire line carrying current i over its whole length. Let us designate this magnetic field distribution by M.

Consider the well-known electric field distribution about an infinitely long two-wire line with voltage e across the line everywhere. Let us designate this electric field distribution by E.

To satisfy Maxwell's equations M and E must fill all space,

* The possibility of expressing electric and magnetic field distributions around a two-wire line having wire resistance in terms of e and i has never been adequately examined from the point of view of Maxwell's equations for the transient condition or state, but it has been examined by G. Mie for the steady direct-current case.

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and it is permissible to think of M as due to the current i in the line and to think of E as due to the charges on the wires.

(*b*) Consider an infinitely thin plane layer of M and E, this layer being at right angles to the line and of thickness dx, the abscissa of the layer being x, the current in the line at x being i, and the voltage across the line at x being e.

Let us imagine a space distribution of magnetic and electric fields built up of an infinity of such layers, i and e being any functions whatever of x. Then it can be shown that this distribution of magnetic and electric fields satisfies Maxwell's equations if

$$\frac{di}{dx} = -C\frac{de}{dt}$$

and

$$\frac{de}{dx} = -L\frac{di}{dt}$$

These are the differential equations of Art. 7, Chapter I for the case in which $R = 0$ and $G = 0$, and these differential equations as applied to a transmission line really refer to the above-described distribution of electric and magnetic fields, or, in other words, *the above-described distribution of electric and magnetic fields is the transmission line solution of Maxwell's equations for the case in which* $R = 0$ *and* $G = 0$.

The electric and magnetic vectors in this field lie everywhere in planes at right angles to the line so that the Poynting vector (the energy stream) is everywhere parallel to the line. There is no flow of energy towards or away from the line anywhere, or, in other words, the line does not radiate energy.

Near the ends of a line the above solution does not hold and the Poynting vector is not everywhere parallel to the line near either end.

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